Further Generalizations of Boolean Rings

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Abstract

McCoy and Montgomery [3] introduced the concept of a \( p \)-ring (\( p \) prime) as a ring \( R \) in which \( x^p = x \) and \( px = 0 \) for all \( x \) in \( R \). Thus, Boolean rings are simply \( 2 \)-rings (\( p = 2 \)). With this as motivation, we define a generalized \( p \)-ring as a ring \( R \) such that \( x^p y - xy^p \in N \) for all \( x, y \in R \setminus (N \cup C) \), and \( px = 0 \) for all \( x \) in \( R \), where \( N \) and \( C \) denote the set of nilpotents and center of \( R \), respectively. We consider the commutativity behavior of generalized \( p \)-rings (\( p \) prime). In particular, we prove that a generalized \( p \)-ring (\( p \) prime) is commutative if and only if its idempotents are central and its nilpotents commute.

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Throughout, \( R \) is a ring, not necessarily with identity, \( N \) is the set of nilpotents, \( C \) is the center, and \( J \) is the Jacobson radical of \( R \). As usual, \([x, y]\) will denote the commutator \( xy - yx \).

Definition 1. A ring \( R \) is called a generalized \( p \)-ring (\( p \) prime) if

\[ x^p y - xy^p \in N \text{ for all } x, y \in R \setminus (N \cup C), \text{ and } px = 0 \text{ for all } x \in R. \]  

(1)

In preparation for the proofs of the main theorems, we need the following three lemmas.

Lemma 1 ([1]). Suppose \( R \) is a ring in which each element \( x \) is central or potent in the sense that \( x^k = x \) for some \( k > 1 \). Then \( R \) is commutative.

Lemma 2. Suppose \( R \) is a ring of prime characteristic \( p \) and with identity, and suppose that all idempotents of \( R \) are central. Then, for all \( b \) in \( R \),

\[ b^p = b \text{ implies } b \text{ is central}. \]  

(2)
Proof. Suppose $b^p = b$. Then $b^{p-1}$ is idempotent, and hence by hypothesis $b^{p-1}$ is central. Therefore, for all $r$ in $R$,

$$b^{p-1}(rb - br) = b^{p-1}(rb) - b^{p-1}r = (rb)b^{p-1} - b^{p-1}r = rb^p - b^p r = rb - br,$$

and hence

$$(b^{p-1} - 1)(rb - br) = 0 \text{ for all } r \in R. \quad (3)$$

Since $R$ is of prime characteristic $p$, an elementary number-theoretic result shows that (3) is equivalent to

$$(b + 1)(b + 2) \ldots (b + (p - 1))(rb - br) = 0, \ (r \in R). \quad (4)$$

Moreover, since $R$ is of prime characteristic $p$,

$$b^p = b \implies (b + 1)^p = b + 1,$$

and hence the above argument may be repeated with $b$ replaced by $b + 1$ throughout to obtain (see (4))

$$(b + 2)(b + 3) \ldots (b + (p - 1))(b + p)(r(b + 1) - (b + 1)r) = 0,$$

and hence (since $b + p = b$)

$$b(b + 2)(b + 3) \ldots (b + (p - 1))(rb - br) = 0. \quad (5)$$

Subtracting (5) from (4), we get

$$1 \cdot (b + 2)(b + 3) \ldots (b + (p - 1))(rb - br) = 0. \quad (6)$$

Repeating this argument, where $b$ is replaced by $(b + 1)$ again throughout, we obtain (see (6))

$$1 \cdot (b + 3)(b + 4) \ldots (b + (p - 1))(b + p)(rb - br) = 0,$$

and hence (since $b + p = b$)

$$1 \cdot b \cdot (b + 3)(b + 4) \ldots (b + (p - 1))(rb - br) = 0. \quad (7)$$

Subtracting (7) from (6), we obtain

$$1 \cdot 2 \cdot (b + 3)(b + 4) \ldots (b + (p - 1))(rb - br) = 0.$$

Continuing this process, we eventually obtain

$$1 \cdot 2 \cdot 3 \ldots (p - 1)(rb - br) = 0. \quad (8)$$

Since $(p - 1)!$ is relatively prime to the prime characteristic $p$ of $R$, we see that (8) implies $rb - br = 0$ for all $r$ in $R$, which proves the lemma. \qed
Lemma 3. Suppose \( R \) is a generalized \( p \)-ring (\( p \) prime), and suppose all idempotents are central. Then, \( N \subseteq J \).

Proof. Let \( a \in N, x \in R \). If \( ax \in N \), then \( ax \) is right quasi regular (r.q.r.). Also, if \( ax \in C \), then \( ax \in N \), and hence again \( ax \) is r.q.r. Moreover, if \((ax)^2 \in N \), then \( ax \in N \), which implies that \( ax \) is r.q.r. Finally, suppose \((ax)^2 \in C \) and \( a^k = 0 \) (since \( a \in N \)). Then, \(((ax)^2)^k = (ax)^2(ax)^2 \ldots (ax)^2 = a^ky = 0 \) for some \( y \in R \). Therefore, \( ax \in N \), and hence again \( ax \) is r.q.r. The only case left to consider is \( ax \notin (N \cup C) \) and \((ax)^2 \notin (N \cup C) \), which implies by (1) that \((ax)^p(ax)^2 - (ax)(ax)^2p \in N \). So \((ax)^{p+2} - (ax)^{2p+1} \in N \). Since \( p+2 \neq 2p+1 \), this implies that

\[
(ax)^q = (ax)^{q+1}g(ax), \text{ for some } g(\lambda) \in \mathbb{Z}[\lambda].
\]

Hence, \( (ax)^q = (ax)^q(axg(ax)) = (ax)^q(axg(ax))^2 = \ldots = (ax)^q(axg(ax))^q \). It is readily verified that \((axg(ax))^q \) is idempotent, and hence

\[
(ax)^q = (ax)^q e, \quad e^2 = e \in aR, \quad e = (axg(ax))^q.
\] (9)

Note that for some \( r \in R \),

\[
e = e^2 = ee = e(ar) = aer \text{ (since } e \text{ is central).}
\]

By re-iterating, we see that

\[
e = aer = a^2er^2 = \ldots = a^ker^k \text{ for all integer } k.
\]

Since \( a \in N \), let \( a^k = 0 \). Then the above equalities imply that \( e = 0 \), and hence by (9), \((ax)^q = 0 \), which implies that \( ax \in N \), and hence \( ax \) is r.q.r. The net result is that \( ax \) is r.q.r. for all \( a \in N, x \in R \), and hence \( a \in J \). This proves the lemma.

We are now in a position to prove our main theorems.

Theorem 1. A generalized \( p \)-ring (\( p \) prime) \( R \) with central nilpotents is commutative.

Proof. Since \( N \subseteq C, N \cup C = C \), and hence (1) becomes

\[
x^py - xy^p \in N \text{ for all } x, y \in R \backslash C.
\] (10)

Suppose \( x \notin C \). We distinguish two cases.

Case 1. \( x^2 \notin C \). Then, \( x \notin C \) and \( x^2 \notin C \), and hence by (10), \( x^p(x^2) - x(x^2)^p \in N \) which implies \( x^{p+2} - x^{2p+1} \in N \). Thus,

\[
(x - x^p)^{p+2} = (x - x^p) \cdot x^{p+1}g(x) = (x^{p+2} - x^{2p+1})g(x) \in N.
\]
So $x - x^p \in N \subseteq C$, and hence $x - x^p \in C$ (if $x^2 \notin C$).

Case 2. $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$). Moreover, $x \notin C$, by hypothesis, which implies by (10),

$$x^p(x - x^2) - x(x - x^2)^p \in N.$$  

Therefore, $x^{p+1} - x^{p+2} - x(x^p - x^{2p}) \in N$ (since $R$ is of characteristic $p$, $p$ prime), and thus $x^{p+2} - x^{2p+1} \in N$, which as we saw in Case 1, implies $x - x^p \in N \subseteq C$. So $x - x^p \in C$. The net result is: $x \notin C$ implies $x - x^p \in C$. Thus, $x - x^p \in C$ for all $x \in R$, which implies by a well known theorem of Herstein [2], that $R$ is commutative. 

\[\square\]

**Corollary 1.** A reduced generalized p-ring is commutative.

**Corollary 2.** A p-ring is commutative.

**Theorem 2.** Suppose $R$ is a generalized p-ring ($p$ prime). Then,

(i) $x \notin C$ implies $x - x^p \in N$.

(ii) $x \notin C$ implies $x^q = x^q e$, $q \geq 1$, $e^2 = e \in x\mathbb{Z}[x]$.

(iii) Every subring and every homomorphic image of a generalized p-ring is also a generalized p-ring.

**Proof.** (i) Suppose $x \notin C$ and $x - x^p \notin N$. Then $x \notin N$, $x \notin C$. We distinguish two cases.

Case 1. $x^2 \notin C$. Then $x^2 \notin N$ and $x^2 \notin C$. Also, $x \notin N$ and $x \notin C$, and hence by (1), $x^p(x^2) - x(x^2)^p \in N$, and hence as in the proof of Theorem 1, $x - x^p \in N$, contradiction.

Case 2. $x^2 \in C$. Then $x - x^2 \notin C$ (since $x \notin C$). If $x - x^2 \in N$, then $(x - x^2) + (x - x^2) + \ldots + x^{p-2}(x - x^2) \in N$, and thus $x - x^p \in N$, contradiction. This contradiction proves that $x - x^2 \notin N$. The net result is: $x - x^2 \notin (N \cup C)$ and $x \notin (N \cup C)$, and hence by (1),

$$x^p(x - x^2) - x(x - x^2)^p \in N.$$  

Thus, as we saw in the proof of Theorem 1, Case 2, $x - x^p \in N$, contradiction. These two contradictions in these two cases prove part (i).

(ii) By part (i), $x - x^p \in N$, and hence $(x - x^p)^q = 0$ for some $q \geq 1$, which implies that $x^q = x^{q+1}g(x)$, $g(x) \in \mathbb{Z}[x]$. Thus, $x^q = x^q(xg(x)) = x^q(xg(x))^2 = \ldots = x^q(xg(x))^q = x^q e$, where $e = (xg(x))^q$ is (as is readily verified) idempotent, proving (ii).

(iii) Follows at once from Definition 1. 

\[\square\]

**Theorem 3.** A generalized p-ring with identity is commutative if and only if $E \subseteq C$ and $N \cap J$ is commutative, where $E$ denotes the set of idempotents.
Proof. Clearly a commutative ring satisfies the two given conditions on $E$ and $N \cap J$. To prove the converse, suppose that $E \subseteq C$ and $N \cap J$ is commutative. By Lemma 3, $N \subseteq J$, and hence $N = N \cap J$ is commutative. Thus,

$$N \text{ is commutative.}$$

(11)

We claim that

$$N \subseteq C.$$ 

(12)

The proof is by contradiction. Suppose not. Then,

for some $a \in N$, $x \in R$, $[a, x] \neq 0$. 

(13)

In view of (11) and (13), $x \notin N$ and $x \notin C$. Also, $[a, x + 1] = [a, x] \neq 0$, and hence, since $a \in N$, $x + 1 \notin (N \cup C)$. Therefore, by (1), $x^p(x+1) - x(x+1)^p \in N$, which implies that $x - x^p \in N$ (since $R$ is of prime characteristic $p$). Thus,

$$x - x^p \in N.$$ 

(14)

Since $x - x^p \in N$, $(x - x^p)^p^k = 0$ for some positive integer $k$, which implies (since $R$ is of prime characteristic $p$)

$$x^p^k = x^{p^k+1}.$$ 

(15)

Moreover, by (14),

$$(x - x^p) + (x - x^p)p + (x - x^p)p^2 + \ldots + (x - x^p)p^{k-1} \in N,$$

and hence $x - x^p^k \in N$. Also, by (15), $(x^p^k)^p = (x^{p^k})$. So

$$x - x^p^k \in N \text{ and } (x^p^k)^p = (x^{p^k}).$$ 

(16)

Moreover, $x^p^k \in C$, by Lemma 2 (see (16)), and hence

$$[a, x] = [a, (x - x^p^k) + x^p^k] = [a, x - x^p^k] = 0,$$

by (16) and (11). Therefore, $[a, x] = 0$, which contradicts (13). This contradiction proves (12). The theorem now follows from (12) and Theorem 1. \qed

**Corollary 3.** A generalized $p$-ring with identity and with central idempotents and commuting nilpotents is commutative.

The next two lemmas will be needed in order to drop the hypothesis that $1 \in R$ in Theorem 3.
Lemma 4. Let $R$ be a generalized $p$-ring, and suppose $\sigma : R \to S$ is a homomorphism of $R$ onto $S$. Then the set $N^*$ of nilpotent elements of $S$ is contained in $\sigma(N) \cup C^*$, where $C^*$ is the center of $S$ and $N$ is the set of nilpotents of $R$.

Proof. Suppose $s \in N^*$, and $s \not\in \sigma(N) \cup C^*$, and suppose $d$ is a preimage of $s$; that is, $\sigma(d) = s$. Then $d \not\in C$ and $d \not\in N$. By Theorem 2(i), $d - dp \in N$. Since $s \in N^*$, $s^q = 0$ for some $q \geq 1$. Note that $(d - dp) + dp - 1(d - dp) + \ldots + (dp - 1)^q(d - dp) = d - dp^{q+p} \in N$, since this is a sum of pairwise commuting nilpotents. This implies that

$$\sigma(d - dp^{q+p}) \in \sigma(N),$$

and hence $s - s^{(p-1)q+p} \in \sigma(N)$, which shows that $s \in \sigma(N)$, (recall that $s^q = 0$), which contradicts the hypothesis that $s \not\in \sigma(N)$. This contradiction proves the lemma.

Lemma 5. Suppose $R$ is a generalized $p$-ring with central idempotents, and suppose $\sigma : R \to R_i$ is a homomorphism of $R$ onto a subdirectly irreducible ring $R_i$. Then either (a) Each element of $R_i$ is central or nilpotent or (b) Each element of $R_i$ is central or nilpotent or a unit in $R_i$.

Proof. By Theorem 2(ii), if $x \in R$, then $x \in C$ or $x^q = x^q e$, $q \geq 1$, $e^2 = e \in xZ[x]$. By hypothesis, $e \in C$, and hence we conclude that

$$x_i \in R_i \text{ implies } x_i \text{ is central or } x_i^q = x_i^q e_i, \ q \geq 1, \ e_i^2 = e_i, \ e_i \in x_iZ[x_i], \ (e_i = \sigma(e)), \ e_i \text{ central.}$$

If $R_i$ does not have an identity then $e_i = 0$, and hence each element of $R_i$ is central or nilpotent. If $R_i$ has an identity, then $e_i = 0$ or $e_i = 1$, and hence each element of $R_i$ is central or nilpotent or a unit in $R_i$, since $e_i \in x_iZ[x_i]$. This proves the lemma.

We are now in a position to drop the hypothesis that $R$ has an identity in Theorem 3.

Theorem 4. Any generalized $p$-ring $R$ is commutative if and only if $E \subseteq C$ and $N \cap J$ is commutative.

Proof. Suppose $E \subseteq C$ and $N \cap J$ is commutative. As is well known, $R \cong$ a subdirect sum of subdirectly irreducible rings $R_i$ ($i \in \Gamma$). Let $\sigma_i : R \to R_i$ be the natural homomorphism. By Lemma 5, either $R_i = N_i \cup C_i$ or $N_i \cup C_i \cup U_i$, where $N_i, C_i, U_i$ denote the set of nilpotents, the center, and the set of units in $R_i$. Moreover, by Lemma 4, $N_i \subseteq \sigma_i(N) \cup C_i$. Furthermore, by Lemma 3, $N \subseteq J$, and hence $N = N \cap J$, and thus $N$ is commutative (since $N \cap J$ is commutative). Since $N_i \subseteq \sigma_i(N) \cup C_i$, it follows that $N_i$ is commutative also. The result is:
(a) \( R_i = N_i \cup C_i \), \( N_i \) commutative, or
(b) \( R_i = N_i \cup C_i \cup U_i \), \( U_i \) = set of units in \( R_i \).

We claim that in case (b), \( U_i \subseteq C_i \). The proof is by contradiction. Suppose not, and let \( u_i \in U_i \) be such that \([u_i, x_i] \neq 0\). Let \( d \) be a preimage of \( u_i \); that is, \( \sigma_i(d) = u_i \). Then \( d \not\in C_i \), and hence by Theorem 2(ii), \( d^q = d^q e \), \( e^2 = e \), \( e \in R \). Therefore, \( u_i^q = u_i^q \sigma_i(e) \), which implies that \( \sigma_i(e) = 1 \), \( e \in C \) (since \( u_i \) is a unit). Hence, \( eR \) is a ring with identity \( e \). Moreover, \( eR \) satisfies all the hypotheses of Theorem 3. (In verifying this, keep in mind that \( N(eR) = eJ(R) \subseteq J(R) \) and \( N(eR) \subseteq N(R) \), and hence \( N(eR) \cap J(eR) \subseteq N(R) \cap J(R) \), which implies that \( N(eR) \cap J(eR) \) is commutative, since, by hypothesis, \( N(R) \cap J(R) = N \cap J \) is commutative.) Therefore, by Theorem 3, \( eR \) is commutative. Let \( x_i, y_i \in R_i \), and let \( \sigma_i(x) = x_i \), \( \sigma_i(y) = y_i \), \( x, y \in R \).

Since \( eR \) is commutative, \( [ex, ey] = 0 \), which implies \( [\sigma_i(ex), \sigma_i(ey)] = 0 \), and hence \( [\sigma_i(e)\sigma_i(x), \sigma_i(e)\sigma_i(y)] = 0 \). So \( [\sigma_i(x), \sigma_i(y)] = 0 \), (since \( \sigma_i(e) = 1 \)), and hence \( [x_i, y_i] = 0 \) for all \( x_i, y_i \in R_i \); that is, \( R_i \) is commutative, a contradiction (since \( [u_i, x_i] \neq 0 \)). This contradiction proves that \( U_i \subseteq C_i \), and hence \( N_i \subseteq C_i \) (since \( a_i \in N_i \) implies \( 1 + a_i \in U_i \subseteq C_i \)), which implies that \( R_i \) is commutative, by Theorem 1 and Theorem 2(iii) (in case (b)).

Returning to case (a), we have \( R_i = N_i \cup C_i \), and \( N_i \) is commutative, which readily implies that \( R_i \) is commutative, and hence \( R \) itself is commutative, which proves the theorem.

Closely related to commutativity is the notion of a ring having a nil commutator ideal. In this connection, we have the following theorem.

**Theorem 5.** Suppose \( R \) is a generalized \( p \)-ring with central idempotents. Then the commutator ideal of \( R \) is nil.

**Proof.** By Lemma 3, \( N \subseteq J \). We now prove that
\[
J \subseteq N \cup C \tag{17}
\]
Suppose \( j \in J \), \( j \not\in C \). Then, by Theorem 2(ii), for some \( q \geq 1 \),
\[
j^q = j^q e, \ e^2 = e \in j\mathbb{Z}[j].
\]
Since \( e \) is an idempotent in \( J \), \( e = 0 \), and hence \( j^q = j^q \cdot 0 = 0 \), which implies that \( j \in N \), proving (17).

Next we prove that
\[
N \text{ is an ideal.} \tag{18}
\]
By (17), and the fact that \( N \subseteq J \), we get \( N \subseteq J \subseteq N \cup C \). Suppose \( a \in N \), \( x \in R \). Then \( a \in J \), \( x \in R \), and hence \( ax \in J \), which implies that \( ax \in N \cup C \).
(by (17)). Hence, $ax \in N$ or $ax \in C$ (which implies that $ax \in N$). So $ax \in N$.

Similarly, $xa \in N$. Next, suppose $a \in N$, $b \in N$. Then $a \in J$, $b \in J$ (since $N \subseteq J$), and hence $a - b \in J \subseteq N \cup C$ (by (17)). So $a - b \in N$ or $a - b \in C$. If $a - b \in C$ then $a$ commutes with $b$, and hence $a - b \in N$ again, which proves (18).

Returning to the ring $R/N$, note that, by Theorem 2(iii), $R/N$ is a generalized $p$-ring, and hence by Theorem 2(i) (applied now to $R/N$), we see that every element of $R/N$ is central or potent. Therefore, by a theorem of Bell (Lemma 1), $R/N$ is commutative, which implies that the commutator ideal of $R$ is nil, and the theorem is proved.

We now consider a certain subclass of generalized $p$-rings ($p$ prime).

**Definition 2.** A $p$-like ring ($p$ prime) is a ring $R$ such that $x^p y = xy^p$ for all $x, y \in R \setminus (N \cup C)$, and $px = 0$ for all $x \in R$.

**Theorem 6.** Suppose $R$ is a $p$-like ring ($p$ prime) with identity. Then $R$ is commutative.

*Proof.* We first prove that

$$N \text{ is commutative.} \tag{19}$$

Suppose that $a \in N$, $u$ is a unit in $R$. Our goal is to prove that

$$[a, u] = 0, \ (a \in N, \ u \ a \ unit). \tag{20}$$

Since $a \in N$, there exists a *minimal* positive integer $\sigma_0$ such that

$$[a^\sigma, u] = 0 \text{ for all } \sigma \geq \sigma_0, \ \sigma_0 \text{ minimal.} \tag{21}$$

We claim that $\sigma_0 = 1$. Suppose not. The *minimality* of $\sigma_0$ in (21), and the fact that $u \not\in C$ (see (20)), show that $1 + a^{\sigma_0 - 1} \not\in (N \cup C)$ and $u \not\in (N \cup C)$, and hence by Definition 2,

$$(1 + a^{\sigma_0 - 1})^p u = (1 + a^{\sigma_0 - 1})u^p,$$

which implies that (since both $1 + a^{\sigma_0 - 1}$ and $u$ are units)

$$(1 + a^{\sigma_0 - 1})^{p-1} = u^{p-1}.$$

Thus,

$$[(1 + a^{\sigma_0 - 1})^{p-1}, u] = 0,$$

which implies (see (21))

$$[1 + (p - 1)a^{\sigma_0 - 1}, u] = 0.$$


Therefore,

\[(p - 1)[a^{\sigma_0 - 1}, u] = 0,\]

which implies that \([a^{\sigma_0 - 1}, u] = 0\) (since \(R\) is of characteristic \(p\)), contradicting the minimality of \(\sigma_0\) (see (21)). This contradiction shows that \(\sigma_0 = 1\), and hence by (21), \([a, u] = 0\), which proves (20). Now, let \(b \in N\). Then, \(1 + b\) is a unit, and hence by (20), \([a, 1 + b] = 0\) (since \(a \in N\)), which implies \([a, b] = 0\) for all \(a, b \in N\), proving (19). Our next goal is to prove that

All idempotents are central. \(\text{(22)}\)

Let \(e^2 = e, x \in R, a = ex - exe\). Suppose \(a \neq 0\). Then, \(a \notin C\) (since \(a \in C\) implies \(ea = ae\), and hence \(a = 0\), contradiction). Since \(a \notin C\) and \(a \in N, 1 + a \notin (N \cup C)\). Moreover, since \(a \neq 0, e \notin (N \cup C)\) either, and hence by Definition 2,

\[e^p(1 + a) = e(1 + a)^p = e(1 + a^p) = e + ea^p = e,\]

which implies that \(e(1 + a) = e\), and thus \(a = 0\), contradiction. Hence, \(a = 0\); that is, \(ex = exe\). Similarly, \(xe = exe\), proving (22). The theorem now follows from (19), (22), and Theorem 4.

Our final result is the following:

**Theorem 7.** Suppose \(R\) is a \(p\)-like ring with central idempotents. Then, \(R\) is isomorphic to a subdirect sum of subdirectly irreducible rings \(R_i\), where \(R_i\) is either nil or commutative.

**Proof.** As is well known, \(R \cong \) a subdirect sum of subdirectly irreducible rings \(R_i\). A careful examination of the proof of Lemma 5 shows that if \(R_i\) does not have an identity, then \(R_i = N_i \cup C_i\), where \(N_i\) and \(C_i\) are the set of nilpotents and center of \(R_i\), respectively, which readily implies \(R_i = N_i\) or \(R_i = C_i\). On the other hand, if \(1 \in R_i\), then by Theorem 6 and Theorem 2(iii), \(R_i\) is commutative, which proves the theorem.

We conclude with the following remarks.

**Remark 1.** Theorems 3 and 4 are not true if in (1) of Definition 1 the two primes used there are different, as can be seen by taking

\[R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a, b, c \in GF(4) \right\}.\]

Note that \(x^7y - xy^7 = 0\) for all \(x, y \in R \setminus (N \cup C)\), and \(2x = 0\) for all \(x\) in \(R\). Moreover, all idempotents of \(R\) are central and \(N\) is commutative. But \(R\) is not commutative.
Remark 2. Theorem 6 is not true if we delete the hypothesis that $1 \in R$, as can be seen by taking

$$R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} : 0, 1 \in GF(2) \right\}. $$

Note that $x^2y - xy^2 = 0$ for all $x, y \in R \setminus (N \cup C)$, and $2x = 0$ for all $x \in R$. But $R$ is not commutative.

Related work appears in [4].

References


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