Approximate Controllability of Semilinear Cascade Systems in $H = L^2(\Omega)$

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Abstract

In this paper we study the interior approximate controllability of the following semilinear cascade systems of $m$ coupled evolution equations in the Hilbert space $H = L^2(\Omega)$

\[
\begin{align*}
\dot{z}_1 &= -Az_1 + 1_\omega u(t) + f_1(t, z_1, u(t)), \quad t \in (0, \tau], \quad \tau > 0 \\
\dot{z}_i &= -Az_i + f_i(t, z_i, z_{i-1}), \quad i = 2, 3, \ldots, m,
\end{align*}
\]

where $\Omega$ is a bounded in $\mathbb{R}^N (N \geq 1)$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$, control $u$ belongs to $L^2(0, \tau; L^2(\Omega))$ and the operator $A : D(A) \subset H \to H$ is an unbounded linear operator with the spectral decomposition $A\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \phi_{j,k} \xi$, $\phi_{j,k} > 0$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \lambda_n \to \infty$ and $\{\phi_{j,k}\}$ is a complete orthonormal set of analytic functions in $H$, and the nonlinear functions $f_i : [0, \tau] \times H \times H \to H$, are smooth enough and there are constant $a_i, c_i \in \mathbb{R}$ and $\xi_i \in Z$, with $c_1 \neq -1$, $a_i < \lambda_1, i = 1, 2, 3, \ldots, m$, such that

\[
\sup_{(t, \zeta, \eta) \in H_\tau} \|f_i(t, \zeta, \eta) - a_i \zeta - c_i \eta - \xi_i\|_H < \infty, \quad i = 1, 2, 3, \ldots, m,
\]

where $c_i \neq 0$, $i = 2, 3, \ldots, m$, $H_\tau = [0, \tau] \times H \times H$. Under these conditions we prove the following statement: The system is approximately controllable on $[0, \tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a $\epsilon$-neighborhood of final state in a prefixed time $\tau$. Our result can be apply to the semilinear $nD$ heat equation, the Ornstein-Uhlenbeck equation, the Laguerre equation, the Jacobi equation, amount others.

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1 Introduction

This paper has been motivated by the work done in [7],[13], [20], [14] and [1], where the authors study, respectively, the controllability of one dimensional coupled of degenerated linear parabolic equations, the null controllability of a cascaded system of linear parabolic-hyperbolic equations, the null controllability of cascade linear systems of $m$ coupled parabolic PDEs by one control force and the exact controllability of a cascade system of conservative linear equations in Hilbert spaces with applications to the Schrödinger equations, the wave equation and the heat equation. As far as we know, the approximate controllability of a semilinear cascade systems of this type has no been studied before. Models of cascade systems can be found in mathematical biology, chemistry, engineering and in a broad variety of physical situations (see [12],[15],[19]). A simple model of this kind of systems appears in mixing problems: Suppose that three tanks connected contains each one 100gls of solution of a certain chemical. Starting at a certain instant a solution of the same chemical, with concentration $u(t)$ lb/gal, is allowed to flows in to the first tank at the rate of $R$ gl/mi. The mixture is drained off at the same rate into the second tank; from the second tank in to the third tank chemical flows at the same rate and the solution flows out of this tank at the same rate. This problem can be formulated as a cascade control system for the amount of chemical in these three tanks at the time $t$. In fact, the three tanks can be denoted respectively by $T_1, T_2$ and $T_3$ and $u(t)$ the concentration of chemical flowing in to $T_1$ acts as a control, $z_i(t)$, $i = 1, 2, 3$ the amount of chemical in $T_i$ at time $t$ and $\frac{z_i(t)}{100}$ the concentration of the chemical in $T_i$ at time $t$. Therefore, the variation of the chemical in each tanks is given by the following cascade system

\[
\begin{align*}
\frac{dz_1(t)}{dt} &= Ru(t) - R\frac{z_1(t)}{100}, \\
\frac{dz_2(t)}{dt} &= R\frac{z_1(t)}{100} - R\frac{z_2(t)}{100}, \\
\frac{dz_3(t)}{dt} &= R\frac{z_2(t)}{100} - R\frac{z_3(t)}{100}.
\end{align*}
\]

(1)

Now, if we put $a = R$ and $b = R/100$, this system can be written as follows

\[
\begin{align*}
z_1'(t) &= au(t) - bz_1(t) \\
z_2'(t) &= bz_1(t) - bz_2(t) \\
z_3'(t) &= bz_2(t) - bz_3(t)
\end{align*}
\]

(2)

and

\[
z' = Az + Bu(t),
\]

(3)
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where

$$A = \begin{pmatrix} -b & 0 & 0 \\ b & -b & 0 \\ 0 & b & -b \end{pmatrix}, \quad B = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

It is well known (see [16]) that the system (3) is controllable if and only if

$$\text{Rank} \left[ B \mid AB \mid A^2B \right] = 3$$

which, doing the respective calculation, is trivially true. In consequence, we have proved that the system (3) is controllable.

According to [1](section 8, pg. 302), nonlinear cascade systems have been studied only in some particular cases of two coupled of parabolic equations (see [9]). In this paper we shall study the interior approximate controllability of the following semilinear cascade system of $m$ coupled evolution equations in the Hilbert space $H = L^2(\Omega)$

$$\begin{array}{l}
\dot{z}_1 = -A z_1 + 1_\omega u(t) + f_1(t, z_1, u(t)), \quad t \in (0, \tau], \quad \tau > 0 \\
\dot{z}_2 = -A z_2 + f_2(t, z_2, z_1), \\
\dot{z}_3 = -A z_3 + f_3(t, z_3, z_2), \\
\vdots \\
\dot{z}_m = -A z_m + f_m(t, z_m, z_{m-1}), \\
\end{array} \quad (4)$$

where $\Omega$ is a bounded in $\mathbb{R}^N (N \geq 1)$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$, control $u$ belongs to $L^2(0, \tau; U)$ with $U = H = L^2(\Omega)$ and the nonlinear functions $f_i : [0, \tau] \times H \times H \rightarrow H$, are smooth enough and there are constants $a_i, c_i \in \mathbb{R}$ and $\xi_i \in Z$, with $c_1 \neq -1$, $a_i < \lambda_1$, $i = 1, 2, 3, \ldots, m$, such that

$$\sup_{(t, \zeta, \eta) \in H_\tau} \| f_i(t, \zeta, \eta) - a_i \zeta - c_i \eta - \xi_i \|_H < \infty, \quad i = 1, 2, 3, \ldots, m, \quad (5)$$

where $i = 2, 3, \ldots, m$, $H_\tau = [0, \tau] \times H \times H$. The operator $A : D(A) \subset H \rightarrow H$ is an unbounded linear operator with the spectral decomposition

$$A \xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \quad (6)$$

with $< \cdot, \cdot >$ denoting an inner product in $H$, and

$$E_j \xi = \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k}.$$
The eigenvalues \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \lambda_n \to \infty\) of \(A\) have finite multiplicity \(\gamma_j\) equal to the dimension of the corresponding eigenspace, and \(\{\phi_{j,k}\}\) is a complete orthonormal set of analytic eigenfunctions (eigenvectors) of \(A\). So, \(\{E_j\}\) is a complete family of orthogonal projections in \(H\) and \(\xi = \sum_{j=1}^{\infty} E_j \xi, \xi \in H\). The operator \(-A\) generates a strongly continuous compact semigroup \(\{T(t)\}\) given by

\[
T(t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi.
\]  

(7)

Without lose of generality, we shall assume that \(m = 3\); that is to say, it is enough to study the particular system of three coupled evolution equations in \(H\)

\[
\begin{aligned}
\dot{z}_1 &= -Az_1 + 1_\omega u(t) + f_1(t, z_1, u(t)), \quad t \in (0, \tau], \quad \tau > 0 \\
\dot{z}_2 &= -Az_2 + f_2(t, z_2, z_1), \\
\dot{z}_3 &= -Az_3 + f_3(t, z_3, z_2).
\end{aligned}
\]  

(8)

Now, we shall describe the strategy of this work:

First, we observe that the hypothesis (9) is equivalent to the existence of constants \(a_i, c_i \in \mathbb{R}\), with \(c_1 \neq -1, a_i < \lambda_1, i = 1, 2, 3\), such that

\[
\sup_{(t, \zeta, \eta) \in H_\tau} \|f_i(t, \zeta, \eta) - a_i \zeta - c_i \eta - \xi_i\|_H < \infty,
\]  

(9)

where \(H_\tau = [0, \tau] \times H \times H\).

Second, we prove that the auxiliary linear system

\[
\begin{aligned}
\dot{z}_1 &= -Az_1 + 1_\omega u(t) + a_1 z_1 + c_1 u(t), \quad t \in (0, \tau], \\
\dot{z}_2 &= -Az_2 + a_2 z_2 + c_2 z_1, \\
\dot{z}_3 &= -Az_3 + a_3 z_3 + c_3 z_2.
\end{aligned}
\]  

(10)

is approximately controllable.

After that, we write the system (4) in the form

\[
\begin{aligned}
\dot{z}_1 &= -Az_1 + 1_\omega u(t) + a_1 z_1 + c_1 u(t) + g_1(t, z_1, u(t)), \quad t \in (0, \tau], \\
\dot{z}_2 &= -Az_2 + a_2 z_2 + c_2 z_1 + g_2(t, z_2, z_1), \\
\dot{z}_3 &= -Az_3 + a_3 z_3 + c_3 z_2 + g_3(t, z_3, z_2).
\end{aligned}
\]  

(11)

where \(g_i(t, \zeta, \eta) = f_i(t, \zeta, \eta) - a_i \zeta - c_i \eta\) are smooth enough and bounded functions.

As an application we can prove the interior approximate controllability of the
following semilinear cascade system of three coupled parabolic equations
\[
\begin{align*}
    v(t, x) &= \Delta v(t, x) + 1_\omega u(t, x) + h_1(t, v, u(t, x)), \quad \text{in} \ (0, \tau] \times \Omega, \\
    w(t, x) &= \Delta w(t, x) + h_2(t, w, v), \quad \text{in} \ (0, \tau] \times \Omega, \\
    y(t, x) &= \Delta y(t, x) + h_3(t, y, w), \quad \text{in} \ (0, \tau] \times \Omega, \\
    v &= w = y = 0, \quad \text{on} \ (0, \tau) \times \partial \Omega, \\
    v(0, x) &= v_0(x), w(0, x) = w_0(x), y(0, x) = y_0(x), \quad x \in \Omega,
\end{align*}
\]  
(12)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \), \( z_0 \in L^2(\Omega) \), \( \omega \) is an open nonempty subset of \( \Omega \), \( 1_\omega \) denotes the characteristic function of the set \( \omega \), the distributed control \( u \) belong to \( L^2([0, \tau]; L^2(\Omega;)) \) and there are constants \( a_i, c_i \in \mathbb{R} \), with \( c_i \neq -1, \ a_i < \lambda_i, \ i = 1, 2, 3 \), such that
\[
\sup_{(t, \zeta, \eta) \in \mathcal{H}_r} \| h_i(t, \zeta, \eta) - a_i \zeta - c_i \eta \| < \infty, \quad (13)
\]

where \( h_r = [0, \tau] \times \mathbb{R} \times \mathbb{R} \).

The technique we use here to prove the controllability of the linear equation (10) is different than the one used in the above reference; it is base on the following basis results from Analytic Functions Theory.

**Theorem 1.1.** (see Theorem 1.23 from [3], pg. 20) Suppose \( \Omega \subset \mathbb{R}^n \) is open, non-empty and connected set, and \( f \) is real analytic function in \( \Omega \) with \( f = 0 \) on a non-empty open subset \( \omega \) of \( \Omega \). Then, \( f = 0 \) in \( \Omega \).

Finally, the approximate controllability of the system (11) follows from the controllability of (10), the compactness of the semigroup generated by the operator \( -A \), the uniform boundedness of the nonlinear terms \( g_i \), and applying Schauder fixed point Theorem.

## 2 Controllability of the Linear Systems

In this section we prove the controllability of the linear system (10); to this end, we notice that for arbitrary \( z_0 = (z_{01}, z_{02}, z_{03})^\top \in Z = H \times H \times H \) and \( u \in L^2([0, \tau]; U) \) the system admits only one mild solution \( z(t) = (z_1(t), z_2(t), z_3(t))^\top \in Z \) given by
\[
\begin{align*}
    z_1(t) &= e^{a_1 t} T(t) z_{01} + \int_0^t e^{a_1 (t-s)} T(t-s) (B_\omega + c_1 I) u(s) ds, \quad (14) \\
    z_2(t) &= e^{a_2 t} T(t) z_{02} + \int_0^t e^{a_2 (t-s)} T(t-s) z_1(s) ds, \quad t \in [0, \tau], \quad (15) \\
    z_3(t) &= e^{a_3 t} T(t) z_{03} + \int_0^t e^{a_3 (t-s)} T(t-s) z_2(s) ds, \quad t \in [0, \tau], \quad (16)
\end{align*}
\]
where \( B_\omega : H \to H \) is the bounded linear operator given by \( B_\omega f = 1_\omega f \) and without lose of generality we assume that \( c_2 = c_3 = 1 \).

**Definition 2.1. (Exact Controllability)** (Fig.1) The system (10) is said to be exactly controllable on \([0, \tau]\) if for every \( z_0, z_1 \in Z \) there exists \( u \in L^2(0, \tau; U) \) such that the mild solution \( z(t) \) of (10) corresponding to \( u \) verifies:

\[
z(0) = z_0 \quad \text{and} \quad z(\tau) = z_1.
\]

**Definition 2.2. (Approximate Controllability)** (Fig.2) The system (10) is said to be approximately controllable on \([0, \tau]\) if for every \( z_0, z_1 \in Z, \varepsilon > 0 \) there exists \( u \in L^2(0, \tau; U) \) such that the mild solution \( z(t) \) of (10) corresponding to \( u \) verifies:

\[
z(0) = z_0 \quad \text{and} \quad \| z(\tau) - z_1 \| < \varepsilon.
\]

**Definition 2.3.** For the system (10) we define the following concept: The controllability map (for \( \tau > 0 \)) \( G : L^2(0, \tau; U) \to Z \) is given by \( Gu = (G_1u, G_2u, G_3u)^\top \) where \( G_1, G_2, G_3 : L^2(0, \tau; U) \to H \) are operators defined as follows

\[
G_1u = \int_0^\tau e^{a_1(\tau-s)} T(\tau-s)(B_\omega + c_1 I)u(s)ds,
\]

\[
G_2u = \int_0^\tau e^{a_2(\tau-s)} T(\tau-s)z_1(s)ds = \int_0^\tau \int_0^s T(\tau-\alpha)(B_\omega + c_1 I)u(\alpha)d\alpha ds,
\]

\[
G_3u = \int_0^\tau e^{a_3(\tau-s)} T(\tau-s)z_2(s)ds = \int_0^\tau \int_0^s \int_0^\alpha T(\tau-\beta)(B_\omega + c_1 I)u(\beta)d\beta d\alpha ds.
\]

Now, using Fubini’s Theorem, these operators can be written as follows:

\[
G_1u = \int_0^\tau e^{a_1(\tau-s)} T(\tau-s)(B_\omega + c_1 I)u(s)ds,
\]

\[
G_2u = \int_0^\tau (\tau-\alpha)e^{a_2(\tau-s)} T(\tau-\alpha)(B_\omega + c_1 I)u(\alpha)d\alpha,
\]

\[
G_3u = \frac{1}{2} \int_0^\tau (\tau-\beta)^2 e^{a_3(\tau-s)} T(\tau-\beta)(B_\omega + c_1 I)u(\beta)d\beta.
\]

whose adjoint operators \( G^*_1, G^*_2, G^*_3 : H \to L^2(0, \tau; U) \) are

\[
(G^*_1z_1)(s) = (B_\omega^* + c_1 I)e^{a_1(\tau-s)} T^*(\tau-s)z_1, \quad \forall s \in [0, \tau],
\]

\[
(G^*_2z_2)(s) = (\tau-s)(B_\omega^* + c_1 I)e^{a_2(\tau-s)} T^*(\tau-s)z_2, \quad \forall s \in [0, \tau],
\]

\[
(G^*_3z_3)(s) = \frac{1}{2}(\tau-s)^2 (B_\omega^* + c_1 I)e^{a_3(\tau-s)} T^*(\tau-s)z_3, \quad \forall s \in [0, \tau].
\]
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Therefore, the adjoint operator \( G^* : Z \to L^2(0, \tau; U) \) of \( G \) is given by the formula

\[
G^* Z = G^* \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = G_1^* z_1 + G_2^* z_2 + G_3^* z_3
\]

(23)

\[
= B_\omega^* T^* (\tau - \cdot) \{ e^{a_1 (\tau - \cdot)} z_1 + (\tau - \cdot) e^{a_2 (\tau - \cdot)} z_2 + \frac{1}{2} (\tau - \cdot)^2 e^{a_3 (\tau - \cdot)} z_3 \}.
\]

The following lemma holds in general for a linear bounded operator \( G : W \to Z \) between Hilbert spaces \( W \) and \( Z \).

**Lemma 2.4.** (see [10], [11], [2] and [17]) The equation (10) is approximately controllable on \([0, \tau]\) if and only if one of the following statements holds:

a) \( \text{Rang}(G) = Z \).

b) \( \text{Ker}(G^*) = \{0\} \).

c) \( \langle GG^* z, z \rangle > 0, \ z \neq 0 \) in \( Z \).

d) \( \lim_{\alpha \to 0^+} \alpha (\alpha I + GG^*)^{-1} z = 0 \).

e) \( \sup_{\alpha > 0} \| \alpha (\alpha I + GG^*)^{-1} \| \leq 1 \).

f) \( (B_\omega^* + c_1 I) T^* (\tau - s) \{ e^{a_1 (\tau - s)} z_1 + (\tau - s) e^{a_2 (\tau - s)} z_2 + \frac{1}{2} (\tau - \cdot)^2 e^{a_3 (\tau - s)} z_3 \} = 0, \ \forall s \in [0, \tau], \Rightarrow z_1 = z_2 = z_3 = 0 \).

g) For all \( z \in Z \) we have \( Gu_\alpha = z - \alpha (\alpha I + GG^*)^{-1} z \), where

\[
u_\alpha = G^* (\alpha I + GG^*)^{-1} z, \quad \alpha \in (0, 1].
\]

So, \( \lim_{\alpha \to 0} Gu = z \) and the error \( E_\alpha z \) of this approximation is given by the formula

\[
E_\alpha z = \alpha (\alpha I + GG^*)^{-1} z, \quad \alpha \in (0, 1].
\]

**Remark 2.5.** The Lemma 2.4 implies that the family of linear operators \( \Gamma_\alpha : Z \to L^2(0, \tau; U) \), defined for \( 0 < \alpha \leq 1 \) by

\[
\Gamma_\alpha z = G^* (\alpha I + GG^*)^{-1} z,
\]

is an approximate inverse for the right of the operator \( G \), in the sense that

\[
\lim_{\alpha \to 0} G \Gamma_\alpha = I.
\]

(25)
Theorem 2.6. The system (10) is approximately controllable on \([0, \tau]\). Moreover, a sequence of controls steering the system (10) from initial state \(z_0\) to an \(\epsilon\) neighborhood of the final state \(z_1\) at time \(\tau > 0\) is given by

\[
u_\alpha(t) = G^*(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0), \quad \alpha \in (0, 1],
\]

and the error of this approximation \(E_\alpha\) is given by

\[
E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0),
\]

where

\[
T_\alpha(t) = \begin{pmatrix}
e^{\alpha_1t}T(t) \\
e^{\alpha_2t}T(t) \\
e^{\alpha_3t}T(t)
\end{pmatrix}.
\]

Proof. We shall apply Lemma 2.4.f to prove the controllability of system (10). To this end, we observe that \(T^*(t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi\) and \(B^*_\omega = B^*_\omega\). Then, for all \(t \in [0, \tau]\) we have that

\[
(G^*z)(t) = (B^*_\omega c_1 I)T^*(\tau-t)\{e^{\alpha_1(t-t)}z_1 + (\tau-t)e^{\alpha_2(t-t)}z_2 + \frac{1}{2}(\tau-t)^2 e^{\alpha_3(t-t)}z_3\} = 0,
\]
i.e.,

\[
\sum_{j=1}^{\infty} e^{-\lambda_j t}\{e^{\alpha_1t}(B^*_\omega + c_1 I)E_j z_1 + te^{\alpha_2t}(B^*_\omega + c_1 I)E_j z_2 + \frac{1}{2}t^2 e^{\alpha_3t}(B^*_\omega + c_1 I)E_j z_3\} = 0.
\]

Now, since \(a_i < \lambda_1\) then \(\lim_{t \to \infty} t^m e^{(a_i-\lambda_1)t} = 0\), and using the ideas of the proof of Lemma 3.14 in [10], pg. 62, we obtain that

\[
e^{\alpha_1t}(B^*_\omega + c_1 I)E_j z_1 + te^{\alpha_2t}(B^*_\omega + c_1 I)E_j z_2 + \frac{1}{2}t^2 e^{\alpha_3t}(B^*_\omega + c_1 I)E_j z_3 = 0, \quad j = 1, 2, 3, \ldots
\]

Since the set of functions \(\{e^{\alpha_1t}, te^{\alpha_2t}, t^2 e^{\alpha_3t}\}\) is linearly independent, we obtain that

\[
(B^*_\omega + c_1 I)E_j z_1 = (B^*_\omega + c_1 I)E_j z_2 = (B^*_\omega + c_1 I)E_j z_3 = 0, \quad j = 1, 2, 3, \ldots
\]

In other words,

\[
\sum_{k=1}^{\gamma_i} <z_i, \phi_{j,k}>(B^*_\omega + c_1 I)\phi_{j,k} = 0, \quad j = 1, 2, 3, \ldots; i = 1, 2, 3.
\]
i.e.,

\[
\sum_{k=1}^{\gamma_i} <z_i, \phi_{j,k}>(1 + c_1)\phi_{j,k}(x) = 0, \quad \forall x \in \omega, j = 1, 2, 3, \ldots; i = 1, 2, 3.
\]
Since, by hypothesis \( \{ \phi_{j,k} \} \) is a complete orthonormal set of analytic functions in \( H \), then applying Theorem 1.1 we obtain that

\[
\sum_{k=1}^{\gamma_j} <z_i, \phi_{j,k}> (1 + c_1) \phi_{j,k}(x) = 0, \quad \forall x \in \Omega, j = 1, 2, 3, \ldots; i = 1, 2, 3,
\]

and consequently

\[(1 + c_1) < z, \phi_{j,k} > = 0, \quad j = 1, 2, \ldots \text{ and } k = 1, 2, \ldots \gamma_j.
\]

Since \( 1 + c_1 \neq 0 \), we get that \( z = 0 \), and the system (10) is approximately controllable on \([0, \tau]\).

Now, given the initial and the final states \( z_0 \) and \( z_1 \), we consider the sequence of controls

\[
u_\alpha(\cdot) = G^*(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0) = G^*(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0), \quad \alpha > 0.
\]

Then,

\[
Gu_\alpha = GG^*(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0) = (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0) = z_1 - e^{\alpha \tau}T(\tau)z_0 - \alpha(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0).
\]

From Lemma 2.4.c we know that

\[
\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1}(z_1 - T_\alpha(\tau)z_0) = 0.
\]

Therefore,

\[
\lim_{\alpha \to 0^+} Gu_\alpha = z_1 - T_\alpha(\tau)z_0.
\]

i.e.,

\[
\begin{align*}
z_{11} &= \lim_{\alpha \to 0^+} \left\{ e^{a_1 \tau}T(\tau)z_{01} + \int_0^\tau e^{a_1(\tau-s)}T(\tau-s)(B_\omega + c_1 I)u_\alpha(s)ds \right\}, \\
z_{12} &= \lim_{\alpha \to 0^+} \left\{ e^{a_2 \tau}T(\tau)z_{02} + \int_0^\tau (\tau-s)e^{a_2(\tau-s)}T(\tau-s)(B_\omega + c_1 I)u_\alpha(s)ds \right\}, \\
z_{13} &= \lim_{\alpha \to 0^+} \left\{ e^{a_3 \tau}T(\tau)z_{03} + \frac{1}{2} \int_0^\tau (\tau-s)^2e^{a_3(\tau-s)}T(\tau-s)(B_\omega + c_1 I)u_\alpha(s)ds \right\},
\end{align*}
\]

where

\[
\begin{pmatrix} z_{11} \\ z_{12} \\ z_{13} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_{01} \\ z_{02} \\ z_{03} \end{pmatrix},
\]

This completes the proof of the Theorem.
3 Controllability of the NonLinear System

In this section we will prove the main result of this paper, the approximate controllability of the semilinear equation (8), which is equivalent to prove the approximate controllability of the system (11); to this end, we notice that this system can be written as first order equation in $Z = H × H × H$

$$\dot{z} = Az + B_\omega u + g(t, z, u(t)), \quad t \in (0, \tau], \quad (26)$$

where $A : D(A) = D(A)^3 \subset Z \to Z$ is an unbounded linear operator given by

$$A = \begin{pmatrix}
(a_1 I - A) & 0 & 0 \\
I & (a_2 I - A) & 0 \\
0 & I & (a_3 I - A)
\end{pmatrix}, \quad (27)$$

and

$$B_\omega = \begin{pmatrix}
B_\omega \\
0 \\
0
\end{pmatrix}, \quad g(t, z, u) = \begin{pmatrix}
g(t, z_1, u) \\
g(t, z_2, z_1) \\
g(t, z_3, z_2)
\end{pmatrix}.$$

Now, using Lemma 2.1 from [18] or Lemma 3.1 from [8], one can get the following representation for this semigroup.

**Proposition 3.1.** The semigroup $\{T(t, A)\}_{t \geq 0}$ generated by the operator $A$ is compact and has the following representation

$$T(t, A)z = \sum_{n,j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0, \quad (28)$$

where $\{P_j\}_{j \geq 0}$ is a complete family of orthogonal projections in the Hilbert space $Z$ given by

$$P_j = \begin{pmatrix}
E_j & 0 & 0 \\
0 & E_j & 0 \\
0 & 0 & E_j
\end{pmatrix}, \quad A_j = R_j P_j, \quad R_j = \begin{pmatrix}
a_1 - \lambda_j & 0 & 0 \\
1 & a_2 - \lambda_j & 0 \\
0 & 1 & a_3 - \lambda_j
\end{pmatrix}, \quad j \geq 1.$$

Next, for arbitrary $z_0 = (z_{01}, z_{02}, z_{03})^\top \in Z$ and $u \in L^2([0, \tau]; U)$, with $U = H$, the initial value problem

$$\begin{align*}
\dot{z} &= Az + B_\omega u + g(t, z, u(t)), \quad t \in (0, \tau], \\
z(0) &= z_0,
\end{align*} \quad (29)$$

admits only one mild solution $z(t) = (z_1(t), z_2(t), z_3(t))^\top \in Z$ given by

$$z_u(t) = T(t, A)z_0 + \int_0^t T(t - s, A)B_\omega u(s) ds$$

$$+ \int_0^t T(t - s, A)g(s, z_u(s), u(s)) ds, \quad t \in [0, \tau]. \quad (30)$$
Definition 3.2. (Approximate Controllability) The system (11) is said to be approximately controllable on \([0, \tau]\) if for every \(z_0, z_1 \in Z, \varepsilon > 0\) there exists \(u \in L^2(0, \tau; U)\) such that the solution \(z(t)\) of (30) corresponding to \(u\) verifies:

\[ z(0) = z_0 \quad \text{and} \quad \|z(\tau) - z_1\| < \varepsilon. \]

Definition 3.3. For the system (11) we define the following concept: The nonlinear controllability map (for \(\tau > 0\)) \(G_g : L^2(0, \tau; U) \rightarrow Z\) is given by

\[ G_g u = \int_0^\tau T(\tau - s, A)B_\omega u(s)ds + \int_0^\tau T(\tau - s, A)g(s, z_u(s), (s))ds = G(u) + H(u), \tag{31} \]

where \(H : L^2(0, \tau; U) \rightarrow Z\) is the nonlinear operator given by

\[ H(u) = \int_0^\tau T(t - s, A)g(s, z_u(s), (s))ds, \quad u \in L^2(0, \tau; U), \tag{32} \]

\(G\) coincides with the operator given by definition 2.3.

The following lemma is trivial:

Lemma 3.4. The equation (11) is approximately controllable on \([0, \tau]\) if and only if

\[ \text{Rang}(G_g) = Z. \]

Definition 3.5. The following equation will be called the controllability equations associated to the nonlinear equation (11)

\[ u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) = G^*(\alpha I + GG^*)^{-1}(z - H(u_\alpha)), \quad (0 < \alpha \leq 1). \tag{33} \]

Now, we are ready to present and prove the main result of this paper, which is approximate controllability of the semilinear diffusion equation (8)

Theorem 3.6. The system (11) is approximately controllable on \([0, \tau]\). Moreover, a sequence of controls steering the system (11) from initial state \(z_0\) to an \(\varepsilon\) neighborhood of the final state \(z_1\) at time \(\tau > 0\) is given by the formula

\[ u_\alpha(t) = B_\omega^*T^*(\tau - t, A)(\alpha I + GG^*)^{-1}(z_1 - T(\tau, A)z_0 - H(u_\alpha)), \]

and the error of this approximation \(E_\alpha\) is given by

\[ E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau, A)z_0 - H(u_\alpha)). \]
Proof. For each $z \in Z$ fixed we shall consider the following family of nonlinear operators $K_\alpha : L^2(0, \tau; U) \to L^2(0, \tau; U)$ given by

$$K_\alpha (u) = \Gamma_\alpha (z - H(u)) = G^* (\alpha I + GG^*)^{-1} (z - H(u)), \quad (0 < \alpha \leq 1). \quad (34)$$

First, we shall prove that for all $\alpha \in (0, 1]$ the operator $K_\alpha$ has a fixed point $u_\alpha$. In fact, since the semigroup $\{T(t, A)\}_{t \geq 0}$ given by (28) is compact (see [5],[6]), then using the result from [4], the smoothness and the boundedness of the non linear term $G$ we obtain that the operator $H$ is compact and the set $\text{Rang}(H)$ is compact.

On the other hand, since $G$ is bounded and $\|T(t, A)\| \leq Re^{\omega t}$, $t \geq 0$, there exists a constant $M > 0$ such that

$$\|H(u)\| \leq M, \quad \forall u \in L^2(0, \tau; U).$$

Then,

$$\|K_\alpha (u)\| \leq \|\Gamma_\alpha \| (\|z\| + M), \quad \forall u \in L^2(0, \tau; U).$$

Therefore, the operator $K_\alpha$ maps the ball $B_r(0) \subset L^2(0, \tau; U)$ of center zero and radius $r \geq \|\Gamma_\alpha \| (\|z\| + M)$ into itself. Hence, applying the Schauder fixed point Theorem we get that the operator $K_\alpha$ has a fixed point $u_\alpha \in B_r(0) \subset L^2(0, \tau; U)$.

Since $\text{Rang}(H)$ is compact, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha (z - H(u)) = G^* (\alpha I + GG^*)^{-1} (z - H(u_\alpha)).$$

Then,

$$Gu_\alpha = GT_\alpha (z - H(u)) = GG^* (\alpha I + GG^*)^{-1} (z - H(u_\alpha))$$

$$= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1} (z - H(u_\alpha))$$

$$= z - H(u_\alpha) - \alpha (\alpha I + GG^*)^{-1} (z - H(u_\alpha))$$

Hence,

$$Gu_\alpha + H(u_\alpha) = z - \alpha (\alpha I + GG^*)^{-1} (z - H(u_\alpha)).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha \to 0} \{-\alpha (\alpha I + GG^*)^{-1} (z - H(u_\alpha))\} = 0$$

From Lemma 2.4.d we get that

$$\lim_{\alpha \to 0} \{-\alpha (\alpha I + GG^*)^{-1} (z - H(u_\alpha))\} = \lim_{\alpha \to 0} \{-\alpha (\alpha I + GG^*)^{-1} H(u_\alpha)\}$$

$$= \lim_{\alpha \to 0} -\alpha (\alpha I + GG^*)^{-1} y - \lim_{\alpha \to 0} -\alpha (\alpha I + GG^*)^{-1} (H(u_\alpha) - y)$$

$$= \lim_{\alpha \to 0} -\alpha (\alpha I + GG^*)^{-1} (H(u_\alpha) - y).$$
On the other hand, from Lemma 2.4.e we get that
\[ \|\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y)\| \leq \|(H(u_\alpha) - y)\|. \]
Therefore, since \(H(u_\alpha)\) converges to \(y\), we get that
\[ \lim_{\alpha \to 0} \{-\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y)\} = 0. \]
Consequently,
\[ \lim_{\alpha \to 0} \{-\alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha))\} = 0. \]
So, putting \(z = z_1 - T(\tau, A)z_0\) and using (30), we obtain the desired result
\[
z_1 = \lim_{\alpha \to 0^+} \{T(\tau, A)z_0 + \int_0^\tau T(\tau - s, A)B_\omega u_\alpha(s)ds
+ \int_0^\tau T(\tau - s, A)g(s, z_{u_\alpha}(s), u_\alpha(s))ds\}.
\]

\[\square\]

4 Applications.

As an application we can prove the interior approximate controllability of the semilinear cascade system of three coupled parabolic equations (12). To this end, we shall need the following Theorem

**Theorem 4.1.** (See [17]) The eigenfunctions of the operator \(-\Delta\) with Dirichlet boundary conditions on \(\Omega\) are real analytic functions in \(\Omega\).

Now, we describe the space in which this problem will be situated as an abstract semilinear cascade system of ordinary differential equations. Let us consider \(H = L^2(\Omega)\) and the linear unbounded operator
\[ A : D(A) \subset H \to H \]
defined by
\[ A\phi = -\Delta \phi, \]
where
\[ D(A) = H^1_0(\Omega) \cap H^2(\Omega). \]
(35)
The operator \(A\) has the following very well known properties: the spectrum of \(A\) consists of only eigenvalues
\[ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty, \]
each one with multiplicity \(\gamma_j\) equal to the dimension of the corresponding eigenspace.

a) There exists a complete orthonormal set \(\{\phi_{j,k}\}\) of analytic eigenfunctions of \(A\).
b) For all $\xi \in D(A)$ we have

$$A\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle \xi, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \quad (36)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H$. The system (12) can be written as an abstract system of ordinary differential equations in the space $H = L^2(\Omega)$

$$\begin{cases}
\dot{v} = -Av + 1_\omega u(t) + h^v_1(t,v,u(t)), & t \in (0, \tau], \\
\dot{w} = -Aw + h^w_c(t,w,v), \\
\dot{y} = -Ay + h^y_3(t,y,w), \\
v(0) = v_0, & w(0) = w_0, \quad y(0) = y_0,
\end{cases} \quad (37)$$

where the control function $u$ belongs to $L^2(0, \tau; H)$ and $h^i_c : [0, \tau] \times Z \times U \to Z$, is defined by $h^i_c(t, \zeta, \eta)(x) = h_i(t, \zeta(x), \eta(x)), \quad \forall x \in \Omega$.

**Theorem 4.2.** For all open non-empty set $\omega \subset \Omega$ and $\tau > 0$ the system $(37)$ is approximately controllable on $[0, \tau]$.

**References**


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