Characterizations of Prime and Weakly Prime Subhypermodules

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Abstract

In this paper, prime and weakly prime subhypermodules of a hypermodule over a hyperring are investigated. We provide some conditions to imply that they are identical. Moreover, prime and weakly prime subhypermodules are characterized in many ways. Finally, characterizations of prime and weakly prime subhypermodules of a multiplication hypermodule are discussed.

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1 Introduction

The study of prime submodules is one interesting topic in module theory. Prime submodules are generalized in many ways, such as weakly prime submodules, semiprime submodules, primary submodules and graded prime submodules, see [2], [5], [7] and [8]. Moreover, the concepts of multiplication modules are studied for characterizing prime submodules, see [1] and [5]. In

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fact, the concepts of modules and prime submodules can be generalized to hypermodules and prime subhypermodules, respectively. P. Corsini introduced, in [4], various definitions of hypermodules such as hypermodules over Krasner hyperrings and feeble hypermodules.

In this paper, we give other definitions of a hyperring and a hypermodule over a hyperring which generalize Krasner hyperrings and hypermodules over Krasner hyperrings. Note also that these definitions generalize modules over rings. We then study prime and weakly prime subhypermodules and show that they are identical under certain conditions. However, our main results are providing various characterizations of prime and weakly prime subhypermodules of a hypermodule over a hyperring in many situations. In addition, we investigate prime and weakly prime subhypermodules of a multiplication hypermodule.

2 Preliminaries

In this section, we give some definitions and properties which will be used in other sections. We denote $\wp^*(H)$ by the collection of all nonempty subsets of a nonempty set $H$. We assume that readers are familiar with canonical hypergroups and their basic properties since they are quite standard see [4]. For a canonical hypergroup $(H,+)$, its scalar identity is denoted by 0 and the inverse of $a \in H$ is denoted by $-a$.

**Definition 2.1.** A **hyperring** is a structure $(R,+,\circ)$ that satisfies the following properties:

(i) $(R,+)$ is a canonical hypergroup with scalar identity 0.

(ii) $(R,\circ)$ is a semihypergroup.

(iii) For all $a,b,c \in R$,
\[ a \circ (b+c) \subseteq a \circ b + a \circ c \text{ and } (b+c) \circ a \subseteq b \circ a + c \circ a. \]

(iv) For all $a,b \in R$, $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If equality holds for both subset relations in (iii), the hyperring is said to be **strongly distributive**. For convenience, we abbreviate a hyperring $(R,+,\circ)$ by a hyperring $R$ and $a \circ b$ by $ab$ for all $a,b \in R$.

**Example 2.2.** Let $(R,+,\cdot)$ be a Krasner hyperring and let $I$ be a hyperideal of $R$. Define $\circ : R \times R \rightarrow \wp^*(R)$ by $a \circ b = a \cdot b + I$ for all $a,b \in R$. Then $(R,+,\circ)$ is a strongly distributive hyperring.
Definition 2.3. Let $R$ be a hyperring. An $R$-hypermodule is a structure $(M, +, \circ)$ such that $(M, +)$ is a canonical hypergroup and $\circ$ is a multivalued scalar operation, i.e., $\circ : R \times M \to \wp^*(M)$, such that for all $a, b \in R$ and $x, y \in M$,

(i) $a \circ (x + y) \subseteq a \circ x + a \circ y$,

(ii) $(a + b) \circ x \subseteq a \circ x + b \circ x$,

(iii) $(ab) \circ x = a \circ (b \circ x)$, and

(iv) $a \circ (-x) = (-a) \circ x = -(a \circ x)$.

If equality holds in (i), the $R$-hypermodule is said to be strongly distributive on the left. Similarly, if equality holds in (ii), the $R$-hypermodule is said to be strongly distributive on the right. Moreover, if equality holds in both (i) and (ii), then the $R$-hypermodule is called strongly distributive. For convenience, we abbreviate an $R$-hypermodule $(M, +, \circ)$ by an $R$-hypermodule $M$ and $a \circ m$ by $am$ for all $a \in R$ and $m \in M$.

This definition generalizes modules over rings. Moreover, this is a generalization of hypermodules over Krasner hyperrings, see [4]. It is easy to see that every hyperring $R$ is an $R$-hypermodule. In fact, there are other types of hyperrings and then of hypermodules depending on the choices of $\oplus, \odot$ of a hyperring $(R, \oplus, \odot)$ and $+, \circ$ of a hypermodule $(M, +, \circ)$.

Example 2.4. Let $M$ be a hypermodule over a Krasner hyperring $R$ and $N$ a subhypermodule of $M$. Define $\circ : R \times M \to \wp^*(M)$ by $a \circ x = ax + N$ for all $a \in R$ and $x \in M$. Then $(M, +, \circ)$ is a strongly distributive $R$-hypermodule.

For a canonical hypergroup $(H, +)$, we define $na$, where $n$ is an integer and $a \in H$, by

$$na = \begin{cases} \{a\} + \{a\} + \cdots + \{a\}, & \text{if } n > 0, \\ \underbrace{\{a\} + \{a\} + \cdots + \{a\}}, & \text{if } n < 0, \\ \underbrace{\{0\}}, & \text{if } n = 0. \end{cases}$$

To avoid any confusion regarding the meanings of $AB$ and $AX$ for any nonempty subsets $A$ and $B$ of a hyperring $R$ and nonempty subsets $X$ of an $R$-hypermodule $M$, we define the following notations.
For nonempty subsets $A$ and $B$ of a hyperring $R$ and nonempty subsets $X$ of an $R$-hypermodule $M,$

\[
AB = \bigcup \{a_ib_i \mid a_i \in A \text{ and } b_i \in B \text{ for all } i\}
\]

\[
[AB] = \bigcup \left\{ \sum_{i=1}^{n} a_ib_i \mid n \in \mathbb{N}, a_i \in A \text{ and } b_i \in B \text{ for all } i \right\}
\]

\[
AX = \bigcup \{a_ix_i \mid a_i \in A \text{ and } x_i \in X \text{ for all } i\}
\]

\[
[AX] = \bigcup \left\{ \sum_{i=1}^{n} a_ix_i \mid n \in \mathbb{N}, a_i \in A \text{ and } x_i \in X \text{ for all } i \right\}.
\]

In particular, let $aB = \{a\}B,$ $Ab = A\{b\},$ $Ax = A\{x\}$ and $aX = \{a\}X$ for all $a, b \in R$ and $x \in M.$

**Definition 2.5.** Let $R$ be a hyperring. A nonempty subset $I$ of $R$ is called a subhyperring of $R$ if $I$ is a hyperring under the same hyperoperations on $R.$ A subhyperring is called a hyperideal of $R$ if $ra \subseteq I$ and $ar \subseteq I$ for all $r \in R$ and $a \in I.$

It is easy to see that $I$ is a hyperideal of $R$ if and only if $[RI] \subseteq I$ and $[IR] \subseteq I.$ Moreover, $I$ is a hyperideal of $R$ if and only if $a - b \subseteq I,$ $ra \subseteq I$ and $ar \subseteq I$ for all $a, b \in I$ and $r \in R.$ In addition, if $I$ and $J$ are hyperideals of a hyperring $R,$ then $I + J$ and $[IJ]$ are hyperideals of $R.$

Next, we introduce the hyperideal generated by a subset of a hyperring.

**Definition 2.6.** Let $A$ be a subset of a hyperring $R.$ Define $\langle A \rangle$ to be the smallest hyperideal of $R$ containing $A.$ The hyperideal $\langle A \rangle$ is called the hyperideal generated by $A.$

Explicit forms of the hyperideal $\langle A \rangle$ for any nonempty subset $A$ of a hyperring $R$ are given as follows. The proofs are straightforward and are omitted.

**Proposition 2.7.** Let $A$ be a nonempty subset of a hyperring $R.$ Then $[RA] + [AR] + [RAR] + [ZA]$ is a hyperideal of $R$ where $[ZA] = \{ \sum_{i=1}^{m} n_ia_i \mid m \in \mathbb{N}, n_i \in \mathbb{Z} \text{ and } a_i \in A \text{ for all } i \}.$ Moreover, $\langle A \rangle = [RA] + [AR] + [RAR] + [ZA].$

**Corollary 2.8.** Let $A$ be a nonempty subset of a hyperring $R.$

(i) If $R$ is commutative, then $\langle A \rangle = [RA] + [ZA].$

(ii) If $a \in Ra$ for all $a \in A,$ then $\langle A \rangle = [RA] + [RAR].$

(iii) If $a \in aR$ for all $a \in A,$ then $\langle A \rangle = [AR] + [RAR].$
(iv) If $R$ is commutative and $a \in Ra$ (equivalently, $a \in aR$) for all $a \in R$, then $\langle A \rangle = [RA] = [AR]$.

The definition of a subhypermodule is now provided. Moreover, some properties of subhypermodules that are parallel to those of hyperideals are given.

**Definition 2.9.** A nonempty subset $N$ of an $R$-hypermodule $M$ is called a subhypermodule of $M$ if $N$ is an $R$-hypermodule under the same hyperoperations.

Using the same ideas as with hyperideals, we obtain that $N$ is a subhypermodule of an $R$-hypermodule $M$ if and only if $[RN] \subseteq N$. Moreover, $N$ is a subhypermodule of $M$ if and only if $x - y \subseteq N$ and $rx \subseteq N$ for all $x, y \in N$ and $r \in R$.

**Proposition 2.10.** Let $M$ be an $R$-hypermodule, $I$ a hyperideal of $R$, $N$ and $K$ subhypermodules of $M$, $a \in R$ and $m \in M$. Then $[aN]$, $[Im]$, $[IN]$, and $N + K$ are subhypermodules of $M$.

We introduce the subhypermodule generated by a nonempty subset of a hypermodule and give some related results.

**Definition 2.11.** Let $X$ be a nonempty subset of a hypermodule $M$. Define $\langle X \rangle$ to be the smallest subhypermodule of $M$ containing $X$. Then the subhypermodule $\langle X \rangle$ is called the subhypermodule generated by $X$.

**Proposition 2.12.** Let $X$ be a nonempty subset of a hypermodule $M$. Then $[RX] + [ZX]$ is a subhypermodule of $M$ where $[ZX] = \{ \sum_{i=1}^{m} n_i x_i \mid m \in \mathbb{N}, n_i \in \mathbb{Z} \text{ and } x_i \in X \text{ for all } i \}$. Moreover, $\langle X \rangle = [RX] + [ZX]$.

**Corollary 2.13.** Let $X$ be a nonempty subset of an $R$-hypermodule $M$ such that $x \in Rx$ for all $x \in M$. Then $\langle X \rangle = [RX]$.

### 3 Prime and Weakly Prime Subhypermodules

In this section, prime and weakly prime subhypermodules of hypermodules are investigated, and many characterizations of them are provided. Throughout this section, let $M$ be an $R$-hypermodule. For nonempty subsets $X$ and $Y$ of $M$, we define $(X : Y) = \{ r \in R \mid rY \subseteq X \}$. It is easy to check that for every subhypermodule $N$ of $M$, the set $(N : M)$ is either the empty set or a hyperideal of $R$ (whereas if $N$ is a submodule of a module $M$ over a ring, $(N : M)$ is always an ideal).

We start with definitions of prime and weakly prime subhypermodules of a hypermodule.
Definition 3.1. Let $R$ be a hyperring and $M$ an $R$-hypermodule. A proper subhypermodule $N$ of $M$ is said to be prime if for every hyperideal $I$ of $R$ and every subhypermodule $D$ of $M$, $[ID] \subseteq N$ implies $I \subseteq (N : M)$ or $D \subseteq N$.

A proper subhypermodule $N$ of $M$ is called weakly prime if for all hyperideals $I$ of $R$ and all subhypermodules $D$ of $M$, $\{0\} \neq [ID] \subseteq N$ implies $I \subseteq (N : M)$ or $D \subseteq N$.

It is clear from the definition that prime subhypermodules are weakly prime subhypermodules.

Example 3.2. Let $R = [0, 1]$. Then $(R, \oplus_{\text{max}}, \cdot)$ is a Krasner hyperring (see [9]), where $\oplus_{\text{max}} : R \times R \to \wp^*(R)$ is the multi-valued function defined by

$$x \oplus_{\text{max}} y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y, \\ [0, x], & \text{if } x = y, \end{cases}$$

and $\cdot$ is the usual multiplication on real numbers. Furthermore, let $K = [0, 0.5]$. Then $K$ is a hyperideal of $R$. It follows from Example 2.2 that $(R, \oplus_{\text{max}}, \cdot)$ is a hyperring. Hence $R$ is an $R$-hypermodule. Moreover, $K$ is a subhypermodule of $R$. Let $R/K = \{r + K \mid r \in R\}$ and define $\oplus : R/K \times R/K \to \wp^*(R/K)$ and $\circ : R \times R/K \to \wp^*(R/K)$ by

$$(a + K) \oplus (b + K) = \{c + K \mid c \in a \oplus_{\text{max}} b\} \quad \text{and} \quad a \circ (b + K) = \{c + K \mid c \in a \circ b\}.$$

It is routine to check that $\oplus$ and $\circ$ are well-defined and that $(R/K, \oplus, \circ)$ is an $R$-hypermodule. It can be shown that $\{K/K\}$ is the zero subhypermodule of $R/K$. Furthermore, it is obvious that $\{K/K\}$ is a weakly prime subhypermodule of $R/K$. But $\{K/K\}$ is not prime, as can be seen by choosing $I = [0, 0.6]$ and $D = [0, 0.5]/K$, so that $I \not\subset \{K/K\} : R/K$ and $D \not\subset \{K/K\}$.

Our first aim is to provide some conditions for weakly prime subhypermodules to be prime. Note that $\{0\}$ is not necessarily a subhypermodule. In the following, we give a necessary and sufficient condition for $\{0\}$ to be a subhypermodule.

Proposition 3.3. Let $M$ be an $R$-hypermodule. Then $\{0\}$ is a subhypermodule of $M$ if and only if there exist a hyperideal $I$ of $R$ and a subhypermodule $N$ of $M$ such that $IN = \{0\}$.

Proof. If $\{0\}$ is a subhypermodule of $M$, then $R\{0\} = \{0\}$. Conversely, assume that there exist a hyperideal $I$ of $R$ and a subhypermodule $N$ of $M$ such that $IN = \{0\}$. Then $\{IN\} = \{0\}$ so that $\{0\}$ is a subhypermodule of $M$. □
Corollary 3.4. If $\{0\}$ is not a subhypermodule of $M$, then prime subhypermodules and weakly prime subhypermodules of $M$ are the same.

Proof. This is an immediate consequence of the previous proposition. 

As a result of Corollary 3.4, we focus on the case that $\{0\}$ is a subhypermodule of $M$.

The following proposition gives another condition which implies that weakly prime subhypermodules are prime subhypermodules.

Proposition 3.5. Let $N$ be a weakly prime subhypermodule of $M$ such that $(N : M) \neq \emptyset$. If $(N : M)N \neq \{0\}$, then $N$ is a prime subhypermodule of $M$.

Proof. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $ID \subseteq N$. If $ID \neq \{0\}$, then we are done. Assume that $ID = \{0\}$.

Case 1: $IN \neq \{0\}$.

Then $IN \subseteq I(D + N) \subseteq ID + IN = \{0\} + IN = IN$. Therefore $\{0\} \neq I(D + N) \subseteq N$. Since $N$ is weakly prime, $I \subseteq (N : M)$ or $D + N \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$.

Case 2: $IN = \{0\}$.

Subcase 2.1 $(N : M)D \neq \{0\}$. Then $(N : M)D \subseteq (I + (N : M))D \subseteq ID + (N : M)D = (N : M)D$. Hence $\{0\} \neq (I + (N : M))D \subseteq N$. Since $N$ is weakly prime, $I + (N : M) \subseteq (N : M)$ or $D \subseteq N$. Hence $I \subseteq (N : M)$ or $D \subseteq N$.

Subcase 2.2 $(N : M)D = \{0\}$. Then $(N : M)N \subseteq (I + (N : M))(D + N) \subseteq ID + IN + (N : M)D + (N : M)N = (N : M)N \subseteq N$. Hence $\{0\} \neq (I + (N : M))(D + N) \subseteq N$. Since $N$ is weakly prime, $I + (N : M) \subseteq (N : M)$ or $D + N \subseteq N$. Therefore $I \subseteq (N : M)$ or $D \subseteq N$. 

Corollary 3.6. Let $N$ be a weakly prime subhypermodule of $M$ which is not prime and $(N : M) \neq \emptyset$. Then $(N : M)N = \{0\}$. In particular, if $I$ is a hyperideal of $R$ such that $I \subseteq (N : M)$, then $IN = \{0\}$.

Proof. It follows from the above proposition that $(N : M)N = \{0\}$. The other result is obvious. 

We are ready to give characterizations of prime subhypermodules of an $R$-hypermodule.

Proposition 3.7. Let $N$ be a proper subhypermodule of $M$. Then $N$ is a prime subhypermodule if and only if $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$ for all hyperideals $I$ of $R$ and all subhypermodules $D$ of $M$. 
First, assume that \( N \) is a prime subhypermodule. Let \( I \) and \( D \) be a hyperideal of \( R \) and a subhypermodule of \( M \), respectively, such that \( ID \subseteq N \). Since \( N \) is a subhypermodule and \([ID]\) is the subhypermodule generated by \( ID \), it follows that \([ID]\) \( \subseteq N \). Hence \( I \subseteq (N:M) \) or \( D \subseteq N \).

The converse follows from the fact that \( ID \subseteq [ID] \).

**Proposition 3.8.** Let \( N \) be a proper subhypermodule of \( M \). Then \( N \) is a prime subhypermodule if and only if \( (N : K) = (N : M) \) for all subhypermodules \( K \) with \( N \subseteq K \subseteq M \).

**Proof.** First, assume that \( N \) is a prime subhypermodule and let \( K \) be a subhypermodule such that \( N \subseteq K \subseteq M \). It is obvious that \( (N : M) \subseteq (N : K) \).

Let \( r \in (N : K) \). Then \( rK \subseteq N \). We show that \( \langle r \rangle K \subseteq N \). Recall that \( \langle r \rangle K = (\{r \} + [r]r + [r]rR + [r]r) \subseteq [rK] + [rRK] + [zrK] \subseteq N \). Then \( \langle r \rangle \subseteq (N : M) \) since \( N \) is prime. Hence \( r \in (N : M) \). This shows \( (N : K) = (N : M) \).

Conversely, assume that \( (N : K) = (N : M) \) for all subhypermodules \( K \) with \( N \subseteq K \subseteq M \). Let \( I \) and \( D \) be a hyperideal of \( R \) and a subhypermodule of \( M \), respectively, such that \( ID \subseteq N \). Suppose \( D \not\subseteq N \). Set \( K = D + N \). Then \( N \subseteq K \subseteq M \) and \( IK = I(D + N) \subseteq ID + IN \subseteq N \). Thus \( I \subseteq (N : K) \).

By assumption, \( I \subseteq (N : M) \). Thus \( N \) is a prime subhypermodule.

Next, weakly prime subhypermodules are characterized.

**Proposition 3.9.** Let \( N \) be a proper subhypermodule of \( M \). Then the following are equivalent.

(i) \( N \) is a weakly prime subhypermodule.

(ii) \( \{0\} \neq ID \subseteq N \) implies \( I \subseteq (N : M) \) or \( D \subseteq N \) for all hyperideals \( I \) of \( R \) and all subhypermodules \( D \) of \( M \).

(iii) \( (N : D) = (N : M) \cup (0 : D) \) for all subhypermodules \( D \) of \( M \) with \( D \not\subseteq N \).

(iv) \( (N : D) = (N : M) \) or \( (N : D) = (0 : D) \) for all subhypermodules \( D \) of \( M \) with \( D \not\subseteq N \).

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( N \) is a weakly prime subhypermodule. Let \( I \) and \( D \) be a hyperideal of \( R \) and a subhypermodule of \( M \), respectively, such that \( \{0\} \neq ID \subseteq N \). Then \( \{0\} \neq ID \subseteq [ID] \subseteq N \). Hence \( I \subseteq (N : M) \) or \( D \subseteq N \).

(ii) \( \Rightarrow \) (iii) Assume that (ii) holds. Let \( D \) be a subhypermodule of \( M \) such that \( D \not\subseteq N \). It is obvious that \( (N : M) \cup (0 : D) \subseteq (N : D) \). Let \( a \in (N : D) \).
Then \( aD \subseteq N \). If \( aD = \{0\} \), then \( a \in (0 : D) \). On the other hand, suppose \( aD \neq \{0\} \). Then \( \{0\} \neq \langle a \rangle D \subseteq N \) so that \( \langle a \rangle \subseteq (N : M) \) or \( D \subseteq N \) from (ii). Consequently, \( a \in \langle a \rangle \subseteq (N : M) \) since \( D \not\subseteq N \).

(iii) \( \Rightarrow \) (iv) This is a well-known property of canonical hypergroups.

(iv) \( \Rightarrow \) (i) Assume that (iv) is valid. Let \( I \) and \( D \) be a hyperideal of \( R \) and a subhypermodule of \( M \), respectively, such that \( \{0\} \neq [ID] \subseteq N \). Suppose that \( D \not\subseteq N \). It follows from (iv) that \( (N : D) = (N : M) \) or \( (N : D) = (0 : D) \). Note that \( I \subseteq (N : D) \) because \( ID \subseteq [ID] \subseteq N \). Thus \( I \subseteq (N : M) \) or \( I \subseteq (0 : D) \). If \( I \subseteq (0 : D) \), then \( ID \subseteq \{0\} \) so that \( [ID] = \{0\} \) leading to a contradiction. Thus \( I \subseteq (N : M) \).

\[ \square \]

**Corollary 3.10.** Let \( N \) be a proper subhypermodule of \( M \). If \( N \) is a weakly prime subhypermodule, then \( (N : \langle m \rangle) = (N : M) \cup (0 : \langle m \rangle) \) for all \( m \in M \setminus N \).

We find that if some conditions on a hyperring \( R \) or on an \( R \)-hypermodule \( M \) are given, then when determining whether a subhypermodule is prime or weakly prime it is enough to consider elements of \( R \) and elements of \( M \) rather than hyperideals of \( R \) and subhypermodules of \( M \). One such condition is that the hyperring \( R \) be commutative, i.e., \( ab = ba \) for all \( a, b \in R \). Another condition is that the hyperring \( R \) satisfy the property, called the **property \( \mathfrak{*}_R \)**, that \( a \in aR \) for all \( a \in R \). One final condition is that the \( R \)-hypermodule \( M \) satisfy the property, called the **property \( \mathfrak{*}_M \)**, that \( m \in Rm \) for all \( m \in M \).

### 3.1 The case \( R \) is commutative

In this subsection, we discuss classifications of prime and weakly prime subhypermodules of an \( R \)-hypermodule when the hyperring \( R \) is commutative.

**Proposition 3.11.** Let \( R \) be a commutative hyperring and \( N \) a proper subhypermodule of an \( R \)-hypermodule \( M \). Then \( N \) is a prime subhypermodule if and only if \( am \subseteq N \) implies \( a \in (N : M) \) or \( m \in N \) for all \( a \in R \) and \( m \in M \).

**Proof.** Suppose first that \( N \) is a prime subhypermodule. Let \( a \in R \) and \( m \in M \) be such that \( am \subseteq N \). Then \( \langle a \rangle \) is a hyperideal of \( R \) and \( \langle m \rangle \) is a subhypermodule of \( M \). Recall from Corollary 2.8 (i) and Proposition 2.12 that \( \langle a \rangle = [Ra] + [Za] \) and \( \langle m \rangle = [Rm] + [Zm] \). Then \( \langle a \rangle \langle m \rangle \subseteq N \) since \( am \subseteq N \) and \( R \) is commutative. Hence \( a \in \langle a \rangle \subseteq (N : M) \) or \( m \in \langle m \rangle \subseteq N \).

Next, assume that \( am \subseteq N \) implies \( a \in (N : M) \) or \( m \in N \) for all \( a \in R \) and \( m \in M \). Let \( I \) and \( D \) be a hyperideal of \( R \) and a subhypermodule of \( M \), respectively, such that \( ID \subseteq N \). Suppose that \( D \not\subseteq N \). Then there exists \( m \in D \setminus N \). Let \( a \in I \). Then \( am \subseteq ID \subseteq N \). By assumption, \( a \in (N : M) \). Thus \( I \subseteq (N : M) \). \[ \square \]
For the corresponding result concerning weakly prime subhypermodules of an $R$-hypermodule $M$, not only the commutativity of $R$ but also the strongly distributivity of $M$ must be required.

**Proposition 3.12.** Let $R$ be a commutative hyperring, $M$ a strongly distributive $R$-hypermodule and $N$ a proper subhypermodule of $M$. Then $N$ is a weakly prime subhypermodule if and only if $N$ is weakly prime.

**Proof.** Assume that $N$ is a weakly prime subhypermodule. Let $a \in R$ and $m \in M$ be such that $\{0\} \neq am \subseteq N$. Thus $\{0\} \neq am \subseteq \langle a \rangle \langle m \rangle$ and $\langle a \rangle \langle m \rangle \subseteq N$. Since $N$ is weakly prime, $a \in \langle N : M \rangle$ or $m \in \langle m \rangle \subseteq N$.

Conversely, assume $\{0\} \neq am \subseteq N$ implies $a \in \langle N : M \rangle$ or $m \in \langle m \rangle \subseteq N$. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $\{0\} \neq ID \subseteq N$. Suppose that $D \nsubseteq N$. Then there exists $x \in D \setminus N$. To show that $I \subseteq (N : M)$, let $a \in I$. If $ax \neq \{0\}$, then $\{0\} \neq ax \subseteq N$, so $a \in (N : M)$ by assumption. Assume that $ax = \{0\}$.

**Case 1:** $aD \neq \{0\}$.

Then there exists $d \in D$ such that $\{0\} \neq ad \subseteq N$. If $d \notin N$, then we are done. Assume that $d \in N$. Since $M$ is strongly distributive on the left, $a(x + d) = ax + ad = ad \neq \{0\}$. Then there exists $l \in x + d$ such that $\{0\} \neq al \subseteq N$. Consequently, $a \in \langle N : M \rangle$ or $l \in N$. If $l \in N$, then $x = l + (-d) \subseteq N$ since $l \in x + d$, which leads to a contradiction. Hence $a \in (N : M)$.

**Case 2:** $aD = \{0\}$.

**Subcase 2.1** $Ix \neq \{0\}$. Then there exists $r \in I$ such that $\{0\} \neq rx \subseteq N$. We obtain that $r \in (N : M)$. Note that $(r + a)x = rx + ax = rx \neq \{0\}$ since $M$ is strongly distributive on the right. Then there exists $p \in r + a$ such that $\{0\} \neq px \subseteq N$ and thus $p \in (N : M)$. Moreover, $a \in (-r) + p \subseteq (N : M)$.

**Subcase 2.2** $Ix = \{0\}$. Since $ID \neq \{0\}$, there exist $b \in I$ and $d \in D$ such that $\{0\} \neq bd \subseteq N$. First, assume that $d \notin N$. Then $b \in (N : M)$. Moreover, by a similar argument as above, $(a + b)d \neq \{0\}$ and thus there exists $p \in a + b$ such that $\{0\} \neq pd \subseteq N$. This leads to the conclusion that $p \in (N : M)$. Hence $a \in p + (-b) \subseteq (N : M)$.

Now, assume that $d \in N$. Then $b(x + d) \neq \{0\}$ and there exists $l \in x + d$ such that $\{0\} \neq bl \subseteq N$. It follows that $b \in (N : M)$ or $l \in N$. If $l \in N$, then $x \in l + (-d) \subseteq N$ which is absurd. Hence $l \notin N$ and $b \in (N : M)$. Moreover, $(a + b)l \neq \{0\}$ and then there exists $p \in a + b$ such that $\{0\} \neq pl \subseteq N$. Again, we can conclude that $a \in p + (-b) \subseteq (N : M)$. 

\[\square\]
3.2 The case $R$ satisfies the property $\ast_R$ or $M$ satisfies the property $\ast_M$

In this subsection, we are interested in studying prime and weakly prime subhypermodules of an $R$-hypermodule $M$ in the case that either $R$ satisfies the property $\ast_R$ or $M$ satisfies the property $\ast_M$. The property $\ast_R$ states that $a \in aR$ for all $a \in R$ while the property $\ast_M$ states that $m \in Rm$ for all $m \in M$. The properties $\ast_R$ and $\ast_M$ generalize the concept that a ring $R$ has an identity and an $R$-module $M$ is unitary, respectively.

**Proposition 3.13.** Let $R$ be a hyperring satisfying the property $\ast_R$ and $N$ a proper subhypermodule of $M$. Then $N$ is a prime subhypermodule if and only if $aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$.

**Proof.** First, assume that $N$ is a prime subhypermodule. Let $a \in R$ and $m \in M$ be such that $aRm \subseteq N$. Corollary 2.8 (ii) and Proposition 2.12 show that $\langle a \rangle = [aR] + [RaR]$ and $\langle m \rangle = [Rm] + [Zm]$. Then $\langle a \rangle \langle m \rangle \subseteq N$ since $aRm \subseteq N$. Hence $a \in \langle a \rangle \subseteq (N : M)$ or $m \in \langle m \rangle \subseteq N$.

Next, assume that $aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $ID \subseteq N$. Suppose that $D \nsubseteq N$. Then there exists $m \in D \setminus N$. To show that $I \subseteq (N : M)$, let $a \in I$. Thus $aR \subseteq IR \subseteq I$. Hence $aRm \subseteq ID \subseteq N$. It follows from the assumption that $a \in (N : M)$. Therefore, $I \subseteq (N : M)$.

In addition, we obtain a similar characterization under the condition that $m \in Rm$ for all $m \in M$.

**Proposition 3.14.** Let $M$ be an $R$-hypermodule satisfying the property $\ast_M$ and $N$ a proper subhypermodule of $M$. Then $N$ is a prime subhypermodule if and only if $aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$.

**Proof.** A similar argument to the proof of the previous proposition yields this result.

The following are characterizations of weakly prime subhypermodules. Although the results are proved in much the same way as the above propositions, the strong distributivity of $M$ is needed.

**Proposition 3.15.** Let $R$ be a hyperring satisfying the property $\ast_R$, $M$ a strongly distributive $R$-hypermodule and $N$ a proper subhypermodule of $M$. Then $N$ is a weakly prime subhypermodule if and only if $\{0\} \neq aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$. 
Proof. Assume that $N$ is a weakly prime subhypermodule. Let $a \in R$ and $m \in M$ be such that $\{0\} \neq aRm \subseteq N$. The result follows from considering the hyperideal $\langle a \rangle$ of $R$ and the subhypermodule $\langle m \rangle$ of $M$.

Now, assume that $\{0\} \neq aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$. Let $I$ be a hyperideal of $R$ and $D$ a subhypermodule of $M$ such that $\{0\} \neq ID \subseteq N$. Suppose that $D \nsubseteq N$. Then there exists $x \in D \setminus N$. To show that $I \subseteq (N : M)$, let $a \in I$. Then $aR \subseteq I$. If $aRx \neq \{0\}$, then we are done. Assume that $aRx = \{0\}$, i.e., $ar = \{0\}$ for all $r \in R$.

Case 1: $aRD \neq \{0\}$.

Then there exists $d \in D$ such that $\{0\} \neq aRd \subseteq N$. We are done if $d \notin N$. Thus assume that $d \in N$. Since $aRD \neq \{0\}$, there exists $r \in R$ such that $ard \neq \{0\}$. Since $M$ is strongly distributive, $ar(x + d) \neq \{0\}$. This leads to the fact that there exists $l \in x + d$ such that $\{0\} \neq aRl \subseteq N$. We have $a \in (N : M)$ or $l \in N$. A contradiction occurs if $l \in N$. Hence $a \in (N : M)$.

Case 2: $aRD = \{0\}$.

Subcase 2.1 $Ix \neq \{0\}$. Then there exists $r \in I$ such that $\{0\} \neq rRx \subseteq ID \subseteq N$. Then $r \in (N : M)$. Moreover, it can be shown similarly that there exists $p \in r + a$ such that $\{0\} \neq pRx \subseteq N$, leading to $p \in (N : M)$. Hence, $a \in (-r) + p \subseteq (N : M)$.

Subcase 2.2 $Ix = \{0\}$. Then there exist $b \in I$ and $d \in D$ such that $\{0\} \neq bRd \subseteq N$. Assume that $d \notin N$. Then $b \in (N : M)$. As before, we obtain that there exists $p \in a + b$ such that $p \in (N : M)$ so that $a \in p + (-b) \subseteq (N : M)$ as desired.

On the other hand, assume that $d \in N$. Then there exists $l \in x + d$ such that $\{0\} \neq bRl \subseteq N$. It can be proved similarly that $l \notin N$ but $b \in (N : M)$. Consequently, there exists $p \in a + b$ such that $p \in (N : M)$ and then $a \in p + (-b) \subseteq (N : M)$.

Proposition 3.16. Let $M$ be a strongly distributive $R$-hypermodule satisfying the property $*_{M}$ and $N$ a proper subhypermodule of $M$. Then $N$ is a weakly prime subhypermodule if and only if $\{0\} \neq aRm \subseteq N$ implies $a \in (N : M)$ or $m \in N$ for all $a \in R$ and $m \in M$.

Proof. This can be obtained similarly.

The next characterization of weakly prime subhypermodules is obtained from Proposition 3.9 but the $R$-hypermodule $M$ needs to be strongly distributive and satisfy the property $*_{M}$.

Proposition 3.17. Let $M$ be a strongly distributive $R$-hypermodule satisfying the property $*_{M}$ and $N$ a proper subhypermodule of $M$. Then the following are equivalent.
Characterizations of prime and weakly prime subhypermodules

(i) $N$ is a weakly prime subhypermodule.

(ii) $(N : Rm) = (N : M) \cup (0 : Rm)$ for all $m \in M \setminus N$.

(iii) $(N : Rm) = (N : M)$ or $(N : Rm) = (0 : Rm)$ for all $m \in M \setminus N$.

Proof. The proofs of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from Proposition 3.9 and the facts for $m \in M \setminus N$ that $[Rm]$ is a subhypermodule of $M$ with $[Rm] \not\subseteq N$, $(N : Rm) = (N : [Rm])$ and $(0 : Rm) = (0 : [Rm])$.

(iii) $\Rightarrow$ (i) Assume that (iii) is valid. Let $a \in R$ and $m \in M$ be such that $\{0\} \neq aRm \subseteq N$. Suppose that $m \notin N$. Note that $a \in (N : Rm)$. By (iii), $a \in (N : M)$ or $a \in (0 : Rm)$. It is not possible that $a \in (0 : Rm)$ since $aRm \neq \{0\}$. Hence $a \in (N : M)$. Therefore, $N$ is a weakly prime subhypermodule by Proposition 3.16.

This subsection concludes with another characterization of weakly prime subhypermodules of a strongly distributive $R$-hypermodule where $R$ is a hyperring satisfying the property $*_R$. Even though the result is similar to the previous proposition, we cannot apply Proposition 3.9 to prove this as before.

**Proposition 3.18.** Let $R$ be a hyperring satisfying the property $*_R$, $M$ a strongly distributive $R$-hypermodule and $N$ a proper subhypermodule of $M$. Then the following are equivalent.

(i) $N$ is a weakly prime subhypermodule.

(ii) $(N : Rm) = (N : M) \cup (0 : Rm)$ for all $m \in M \setminus N$.

(iii) $(N : Rm) = (N : M)$ or $(N : Rm) = (0 : Rm)$ for all $m \in M \setminus N$.

Proof. (i) $\Rightarrow$ (ii) Assume that $N$ is a weakly prime subhypermodule. Let $m \in M \setminus N$. It is obvious that $(N : M) \cup (0 : Rm) \subseteq (N : Rm)$. Let $a \in (N : Rm)$. Thus $aRm \subseteq N$. If $aRm = \{0\}$, then $a \in (0 : Rm)$. If $aRm \neq \{0\}$, then $a \in (N : M)$ by the assumption together with Proposition 3.15.

(ii) $\Rightarrow$ (iii) This is a well-known property of canonical hypergroups.

(iii) $\Rightarrow$ (i) This is similar to the proof (iii) $\Rightarrow$ (i) of Proposition 3.17 but Proposition 3.15 must be applied instead.

4 Prime and Weakly Prime Subhypermodules of Multiplication Hypermodules

This section discusses another main result of this paper. Our purpose here is to characterize prime and weakly prime subhypermodules in the context
that an $R$-hypermodule is a multiplication $R$-hypermodule. We first give the definition of a multiplication $R$-hypermodule and provide some properties used for proving our main results.

**Definition 4.1.** Let $M$ be an $R$-hypermodule. Then $M$ is called a multiplication $R$-hypermodule if for each subhypermodule $N$ of $M$, there exists a hyperideal $I$ of $R$ such that $N = [IM]$.

Recall that, in general, if $N$ is a subhypermodule of an $R$-hypermodule $M$, then $(N : M)$ may be empty. However, if $M$ is a multiplication $R$-hypermodule, then, for any subhypermodule $N$ of $M$, one can show that $(N : M)$ is nonempty, and in fact it is a hyperideal of $R$. Moreover, any subhypermodule $N$ of a multiplication $R$-hypermodule $M$ can be written as $[(N : M)M]$; consequently, $M = [RM]$ and $N = M$ if and only if $(N : M) = R$.

For modules over a ring, if $N$ is a prime submodule of an $R$-module $M$, then $(N : M)$ is a prime ideal of $R$. This is also valid for hypermodules over a hyperring. First, we need the definition of a prime hyperideal.

**Definition 4.2.** Let $R$ be a hyperring. A proper hyperideal $P$ of $R$ is said to be prime if for all hyperideals $I$ and $J$ of $R$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

**Proposition 4.3.** Let $N$ be a subhypermodule of an $R$-hypermodule $M$ such that $(N : M) \neq \emptyset$ and $(N : M) \neq R$. If $N$ is a prime subhypermodule, then $(N : M)$ is a prime hyperideal of $R$.

**Proposition 4.4.** Let $M$ be a multiplication $R$-hypermodule and $N$ a subhypermodule of $M$. Then $N$ is a prime subhypermodule if and only if $(N : M)$ is a prime hyperideal of $R$.

**Proposition 4.4.** Let $M$ be a multiplication $R$-hypermodule and $N$ a subhypermodule of $M$. Then $N$ is a prime subhypermodule if and only if $(N : M)$ is a prime hyperideal of $R$. 

**Proof.** The necessary part is clear. Next, assume that $(N : M)$ is a prime hyperideal of $R$. Then $(N : M) \neq R$ so that $N \neq M$. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $ID \subseteq N$. Since $M$ is a multiplication $R$-hypermodule, $D = [JM]$ for some hyperideal $J$
of $R$. Thus $(IJ)M = I(JM) \subseteq I[JM] = ID \subseteq N$. This shows that $IJ \subseteq (N : M)$. Since $(N : M)$ is a prime hyperideal of $R$, either $I \subseteq (N : M)$ or $J \subseteq (N : M)$. Then $I \subseteq (N : M)$ or $JM \subseteq N$. Thus $I \subseteq (N : M)$ or $D \subseteq N$. Hence $N$ is a prime subhypermodule. 

The product of subhypermodules of a multiplication $R$-hypermodule is defined analogously to the product of submodules of a multiplication module over a ring.

**Definition 4.5.** Let $R$ be a commutative hyperring and $M$ be a multiplication $R$-hypermodule. For subhypermodules $N$ and $K$ of $M$, the **product of $N$ and $K$**, denoted by $NK$, is defined as $NK = [(IJ)M]$ where $N = [IM]$ and $K = [JM]$ for some hyperideals $I$ and $J$ of $R$.

Moreover, we define $mn' = \langle m \rangle \langle m' \rangle$, $mN = \langle m \rangle N$ and $Nm = N \langle m \rangle$ for any $m, m' \in M$.

It is straightforward to show that the product of subhypermodules is well-defined. We note here that products of subhypermodules of a multiplication $R$-hypermodule require the commutativity of the hyperring $R$. As a result, for the rest of this paper let $R$ be a commutative hyperring.

**Proposition 4.6.** Let $N$ be a proper subhypermodule of a multiplication $R$-hypermodule $M$. Then $N$ is a prime subhypermodule if and only if $PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules $P$ and $K$ of $M$.

**Proof.** First, assume that $N$ is a prime subhypermodule. Let $P$ and $K$ be subhypermodules of $M$ such that $PK \subseteq N$. Suppose $P = [IM]$ and $K = [JM]$ for some hyperideals $I$ and $J$ of $R$. Then $I[JM] \subseteq [(IJ)M] \subseteq [(IJ)M] = PK \subseteq N$. Since $N$ is prime, $I \subseteq (N : M)$ or $JM \subseteq N$. Hence $IM \subseteq N$ or $K \subseteq N$. Thus $P = [IM] \subseteq N$ or $K \subseteq N$.

Conversely, assume that $PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules $P$ and $K$ of $M$. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $[ID] \subseteq N$. Suppose $D = [JM]$ for some hyperideal $J$ of $R$. Set $P = [IM]$. Then $P$ and $D$ are subhypermodules of $M$ such that $PD = [(IJ)M] = [I[JM]] = [ID] \subseteq N$. By assumption, $P \subseteq N$ or $D \subseteq N$. Thus $IM \subseteq [IM] = P \subseteq N$ or $D \subseteq N$. Therefore $I \subseteq (N : M)$ or $D \subseteq N$. Hence $N$ is a prime subhypermodule. 

To obtain another classification of prime subhypermodules of a multiplication $R$-hypermodule $M$, the following lemma is required.

**Lemma 4.7.** Let $M$ be a multiplication $R$-hypermodule and $N, P$ and $K$ subhypermodules of $M$. Then the following hold.
A. Siraworakun, S. Pianskool and M. Hall

(i) \( PK \subseteq N \) if and only if \( pK \subseteq N \) for all \( p \in P \).

(ii) \( PK \subseteq N \) if and only if \( Pk \subseteq N \) for all \( k \in K \).

(iii) \( PK \subseteq N \) if and only if \( pk \subseteq N \) for all \( p \in P \) and \( k \in K \).

Proof. (i) First, assume that \( PK \subseteq N \). Then \( [(P : M)(K : M)]M \subseteq N \). Let \( p \in P \). Thus \( \langle p \rangle = [I_pM] \) for some hyperideal \( I_p \) of \( R \). Note that \( I_pM \subseteq P \) so that \( I_p \subseteq (P : M) \). Hence

\[
pK = \langle p \rangle K = \left[ I_p(K : M) \right] M \subseteq \left[ (P : M)(K : M) \right] M \subseteq N.
\]

Conversely, assume that \( pK \subseteq N \) for all \( p \in P \). Then for each \( p \in P \) there exists a hyperideal \( I_p \) of \( R \) such that \( \langle p \rangle = [I_pM] \), so that \( \left[ I_p(K : M) \right] M = pK \subseteq N \) for all \( p \in P \). Since \( P = \sum_{p \in P} \langle p \rangle \), we obtain that

\[
PK = \left( \sum_{p \in P} [I_pM] \right) K = \left[ \left( \sum_{p \in P} I_p \right) M \right] [(K : M)M]
\]

\[
= \left[ \left( \sum_{p \in P} I_p \right) (K : M) \right] M
\]

\[
= \sum_{p \in P} \left[ I_p(K : M) \right] M \subseteq N.
\]

(ii) The proof is similar to (i).

(iii) This follows from (i) and (ii).

Proposition 4.8. Let \( N \) be a proper subhypermodule of a multiplication \( R \)-hypermodule \( M \). Then \( N \) is a prime subhypermodule if and only if \( mm' \subseteq N \) implies \( m \in N \) or \( m' \in N \) for all \( m, m' \in M \).

Proof. The necessary part is obtained from Proposition 4.6 and the definition of the product \( mm' \).

Conversely, assume that \( mm' \subseteq N \) implies \( m \in N \) or \( m' \in N \) for all \( m, m' \in M \). Let \( P \) and \( K \) be subhypermodules of \( M \) such that \( PK \subseteq N \) and \( K \nsubseteq N \). Then there exists \( k \in K \setminus N \). To show that \( P \subseteq N \), let \( p \in P \). Lemma 4.7 (iii) yields \( pk \subseteq N \) hence \( p \in N \) by assumption. This shows that \( P \subseteq N \). We conclude that \( N \) is a prime subhypermodule.

Finally, weakly prime subhypermodules of a multiplication \( R \)-hypermodule are characterized.
Proposition 4.9. Let $N$ be a proper subhypermodule of a multiplication $R$-hypermodule $M$. Then $N$ is a weakly prime subhypermodule if and only if \( \{0\} \neq PK \subseteq N \) implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules $P$ and $K$ of $M$.

Proof. First, assume that $N$ is a weakly prime subhypermodule. If $N$ is a prime subhypermodule, then we are done. Suppose that $N$ is not a prime subhypermodule. Let $P$ and $K$ be subhypermodules of $M$ such that $\{0\} \neq PK \subseteq N$ and assume for a contradiction that $P \nsubseteq N$ and $K \nsubseteq N$. We show that $PK = \{0\}$ by applying Lemma 4.7. We claim that $pK = \{0\}$ for all $p \in P \setminus N$. Let $p \in P \setminus N$, $\langle p \rangle = [I_pM]$ and $K = [IM]$ for some hyperideals $I_p$ and $I$ of $R$. Moreover, let $P = \sum_{l \in P} \langle l \rangle$. Then $P = \left( \sum_{l \in P} I_l \right)M$ where $I_l$ is a hyperideal of $R$ such that $\langle l \rangle = [I_lM]$ for all $l \in P$. Hence $pK = \langle p \rangle K = [I_pI]M \subseteq \left[ \left( \sum_{l \in P} I_l \right)I \right]M = PK \subseteq N$. Therefore, $I(p) = I[I_pM] \subseteq [I[I_pM]] = [I[I_p]M] \subseteq N$, i.e., $I \subseteq (N : \langle p \rangle)$. Recall from Corollary 3.10 that $(N : \langle p \rangle) = (N : M) \cup (0 : \langle p \rangle)$. As a result, $I \subseteq (N : M)$ or $I \subseteq (0 : \langle p \rangle)$. Thus $IM \subseteq N$ or $I(p) = \{0\}$ so that $K = [IM] \subseteq N$ or $I(p) = \{0\}$. Since $K \nsubseteq N$, it forces that $I(p) = \{0\}$. Therefore $pK = [I[I_p]M] = [I[I_p]M] = [I[I_p]M] = \{0\}$ as claimed. Similarly, $P_k = \{0\}$ for all $k \in K \cap N$. It remains to show that $pk = \{0\}$ for all $p \in P \cap N$ and $k \in K \cap N$. Let $p \in P \cap N$ and $k \in K \cap N$. Thus, $pk = \langle p \rangle \langle k \rangle \subseteq NN = ((N : M)N) = \{0\}$ by Corollary 3.6. We then can conclude that $PK = \{0\}$ leading to a contradiction. Therefore, $P \subseteq N$ or $K \subseteq N$.

Conversely, assume that $\{0\} \neq PK \subseteq N$ implies $P \subseteq N$ or $K \subseteq N$ for all subhypermodules $P$ and $K$ of $M$. Let $I$ and $D$ be a hyperideal of $R$ and a subhypermodule of $M$, respectively, such that $\{0\} \neq ID \subseteq N$. Let $P = [IM]$. Then $PD = [I(D : M)M] = [I(D : M)M] = [ID] \subseteq N$. Thus $\{0\} \neq PD \subseteq N$ and hence $P \subseteq N$ or $D \subseteq N$. Consequently, $I \subseteq (N : M)$ or $D \subseteq N$. This shows that $N$ is a weakly prime subhypermodule. \qed

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