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Upper and Lower (α, β)- Intuitionistic Fuzzy Set

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Abstract

The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. Using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets. We introduce the concept of upper and lower of an (α, β)-intuitionistic fuzzy set, where α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ with $\alpha \neq \in \wedge q$, and some interesting properties are investigated.

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1 Introduction

The theory of fuzzy sets was first developed by Zadeh [18] and has been applied to many branches in mathematics (see, for example, [1, 6, 14]).

The notion of intuitionistic fuzzy sets introduced by Atanassov [2, 6] as a generalization of the notion of fuzzy sets. As the basis for the study of intuitionistic fuzzy set theory, many operations and relations over intuitionistic fuzzy sets were introduced [3-6]. In [7] Biswas applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group.

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [15], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [9, 10] gave the concepts of (α, β)-fuzzy subgroups by using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" between a fuzzy point and a fuzzy subgroup, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [11] $(\in, \in \vee q)$-fuzzy subrings and ideals defined. In [8] Bhakat defined $(\in \vee q)$-level subsets of a fuzzy set. In [16] Shabir, Jun et al.

studied characterizations of regular semigroups by (α, β)-fuzzy ideals. In [17] Yuan, Li et al. redefined (α, β)-intuitionistic fuzzy subgroups.

The paper is organized as follows: in Section 2 some fundamental definitions on fuzzy sets and IF sets are explored, and in Section 3 we present some fundamental definitions on upper and lower of an (α, β)-intuitionistic fuzzy set, and establish some useful theorems.

2 Preliminaries

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [18] in 1965. Let X be a non-empty set. A mapping $\mu : X \longrightarrow [0;1]$ is called a *fuzzy set* in X. The *complement* of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

For any $t \in [0, 1]$ and fuzzy set μ of X, the set

$$U(\mu, t) = \{x \in X | \mu(x) \geq t\} \ \ \text{(respectively, } L(\mu, t) = \{x \in X | \mu(x) \leq t\}),$$

is called an *upper* (respectively, *lower*) *t-level cut* of μ.

Definition 2.1. An *intuitionistic fuzzy set* (IFS for short) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\},$$

where the functions $\mu_A : X \longrightarrow [0;1]$ and $\lambda_A : X \longrightarrow [0;1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ with respect to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$ (see [2, 4]). For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the IFS $A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$. Denote by $IFS(X)$ the set of all intuitionistic fuzzy sets in X.

Definition 2.2. [2] Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in X. Then
(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(3) $A^c = \{(x, \lambda_A(x), \mu_A(x)) | x \in X\}$,
(4) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) | x \in X\}$,
(5) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) | x \in X\}$.

3 Upper and Lower (α, β)- Intuitionistic Fuzzy Set

Definition 3.1. A fuzzy subset μ of a universe X is a function from X into the unit closed interval $[0, 1]$, i.e. $\mu : X \longrightarrow [0, 1]$ (see [18]). A fuzzy subset μ

in a universe X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set μ written $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t \in \vee q \mu$ (resp. $x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$).

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ with $\alpha \neq \in \wedge q$, which was introduced by Bhakat and Das [10].

Definition 3.2. Let $t \in (0,1]$ and μ is a fuzzy set in X. We defined

$$U(\alpha\mu, t) = \{x \in X |\ x_t \alpha \mu\}, \quad L(\in \mu, t) = \{x \in X |\ \mu(x) \leq t\},$$

$$L(q\mu, t) = \{x \in X |\ \mu(x) + t \leq 1\}),$$

$$L(\in \vee q\mu, t) = \{x \in X |\ \mu(x) + t \leq 1 \text{ or } \mu(x) \leq t\},$$

where $\alpha \in \{\in, q, \in \vee q\}$. Then,
(UL_1) the set $U(\in \mu, t)$ and $L(\in \mu, t)$ is called an *upper* and *lower t-level cut* of $\in \mu$, respectively,
(UL_2) the set $U(q\mu, t)$ and $L(q\mu, t)$ is called an *upper* and *lower t-level cut* of $q\mu$, respectively,
(UL_3) the set $U(\in \vee q\mu, t)$ and $L(\in \vee q\mu, t)$ is called an *upper* and *lower t-level cut* of $\in \vee q\mu$, respectively.

It is clear that $U(\in \mu, t) = U(\mu, t)$ and $L(\in \mu, t) = L(\mu, t)$.

Theorem 3.3. Let μ is a fuzzy set in X, then for all $t \in (0,1]$ we have

(1) $U(\in \vee q\mu_A, t) = U(\in \mu_A, t) \bigcup U(q\mu_A, t)$,

(2) $L(\in \vee q\mu_A, t) = L(\in \mu_A, t) \bigcup L(q\mu_A, t)$.

Proof. (1) Let μ is a fuzzy set in X and $t \in (0,1]$, then for all $x \in X$ we have

$$\begin{aligned} x \in U(\in \vee q\mu_A, t) &\iff x_t \in \vee q\mu_A \\ &\iff x_t \in \mu_A \text{ or } x_t q \mu_A \\ &\iff x \in U(\in \mu_A, t) \text{ or } x \in U(q\mu_A, t) \\ &\iff x \in (U(\in \mu_A, t) \bigcup U(q\mu_A, t)). \end{aligned}$$

Thus $U(\in \vee q\mu_A, t) = U(\in \mu_A, t) \bigcup U(q\mu_A, t)$.

(2) Let μ is a fuzzy set in X and $t \in (0,1]$, then for all $x \in X$ we have

$$\begin{aligned} x \in L(\in \vee q\mu_A, t) &\iff \mu(x) + t \leq 1 \text{ or } \mu(x) \leq t \\ &\iff x \in L(\in \mu_A, t) \text{ or } x \in L(q\mu_A, t) \\ &\iff x \in (L(\in \mu_A, t) \bigcup L(q\mu_A, t)). \end{aligned}$$

Thus $L(\in \vee q\mu_A, t) = L(\in \mu_A, t) \bigcup L(q\mu_A, t)$.

□

Corollary 3.4. [12] For any fuzzy subset λ of X and $t \in (0,1]$, we consider two subsets:

$$Q(\lambda, t) = \{x \in X |\ x_t q\lambda\} \text{ and } [\lambda]_t = \{x \in X |\ x_t \in \vee q\lambda\}.$$

Then $[\lambda]_t = U(\lambda, t) \bigcup Q(\lambda, t)$.

Theorem 3.5. Let μ is a fuzzy set in X, we have

(1) Let $t \in (0, 0.5]$, then $U(\in \vee q\mu_A, t) = U(\in \mu_A, t)$,

(2) Let $t \in (0.5, 1]$, then $U(\in \vee q\mu_A, t) = U(q\mu_A, t)$.

Proof. (1) Let $t \in (0, 0.5]$, then $1 - t \in [0.5, 1)$. Thus $t \leq 1 - t$.
By Theorem 3.3, it is clear that $U(\in \mu_A, t) \subseteq U(\in \vee q\mu_A, t)$. Let $x \notin U(\in \mu_A, t)$, then $\mu_A(x) < t$ and so $\mu_A(x) < 1 - t$. This shows that $x \notin U(q\mu_A, t)$, and thus $x \notin (U(\in \mu_A, t) \bigcup U(q\mu_A, t))$. Therefore $U(\in \mu_A, t) \supseteq U(\in \vee q\mu_A, t)$. Hence $U(\in \mu_A, t) = U(\in \vee q\mu_A, t)$.

(2) Let $t \in (0.5, 1]$, then $1 - t \in [0, 0.5)$. Thus $1 - t < t$.
By Theorem 3.3, we have $U(q\mu_A, t) \subseteq U(\in \vee q\mu_A, t)$. Let $x \notin U(q\mu_A, t)$, then $\mu_A(x) + t \leq 1$ and so $\mu_A(x) \leq 1 - t < t$. This shows that $x \notin U(\in \mu_A, t)$, and thus $x \notin (U(\in \mu_A, t) \bigcup U(q\mu_A, t))$. Therefore $U(q\mu_A, t) \supseteq U(\in \vee q\mu_A, t)$. Hence $U(q\mu_A, t) = U(\in \vee q\mu_A, t)$.

□

Corollary 3.6. [13] Every fuzzy subset λ of X satisfies the following assertion:

$$t \in (0, 0.5] \Longrightarrow [\lambda]_t = U(\lambda, t).$$

Theorem 3.7. Let μ is a fuzzy set in X, we have

(1) Let $t \in (0, 0.5]$, then $L(\in \vee q\mu_A, t) = L(\in \mu_A, t)$,

(2) Let $t \in (0.5, 1]$, then $L(\in \vee q\mu_A, t) = L(q\mu_A, t)$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Theorem 3.8. Let $A=(\mu_A,\lambda_A), B=(\mu_B,\lambda_B)\in IFS(X)$. Then

(1) $A\subseteq B \Longrightarrow U(\alpha\mu_A,t)\subseteq U(\alpha\mu_B,t)$,

(2) $A\subseteq B \Longrightarrow L(\alpha\lambda_A,t)\subseteq L(\alpha\lambda_B,t)$,

(3) $A=B \iff U(\alpha\mu_A,t)=U(\alpha\mu_B,t)$ and $L(\alpha\lambda_A,t)=L(\alpha\lambda_B,t)$,

where $\alpha\in\{\in,q,\in\vee q\}$.

Proof. (1) Let $A\subseteq B$. Then $\mu_A(x)\leq\mu_B(x)$ and $\lambda_A(x)\geq\lambda_B(x)$ for all $x\in X$. Thus

(a) If $x\in U(\in\mu_A,t)$, then $t\leq\mu_A(x)\leq\mu_B(x)$. This shows that $x\in U(\in\mu_B,t)$, i.e., $U(\in\mu_A,t)\subseteq U(\in\mu_B,t)$.

(b) If $x\in U(q\mu_A,t)$, then $1<\mu_A(x)+t\leq\mu_B(x)+t$. This shows that $x\in U(q\mu_B,t)$, i.e., $U(q\mu_A,t)\subseteq U(q\mu_B,t)$.

(c) If $x\in U(\in\vee q\mu_A,t)$, then by (a) and (b), we have $x\in U(\in\vee q\mu_B,t)$.

(2) The proof is similar to the proof of (1).

(3) The proof is straightforward.

□

Theorem 3.9. Let $A=(\mu_A,\lambda_A)\in IFS(X)$. Then, we have

(1) If $t,\acute{t}\in(0,1]$, $t\leq\acute{t}$. Then $U(\in\mu_A,\acute{t})\subseteq U(\in\mu_A,t)$,

(2) If $t,\acute{t}\in(0,1]$, $t\leq\acute{t}$. Then $U(q\mu_A,\acute{t})\supseteq U(q\mu_A,t)$,

(3) If $t,\acute{t}\in(0,0.5]$, $t\leq\acute{t}$. Then $U(\in\vee q\mu_A,\acute{t})\subseteq U(\in\vee q\mu_A,t)$,

(4) If $t,\acute{t}\in(0.5,1]$, $t\leq\acute{t}$. Then $U(\in\vee q\mu_A,\acute{t})\supseteq U(\in\vee q\mu_A,t)$,

(5) If $t,\acute{t}\in(0,1]$, $t\leq\acute{t}$. Then $L(\in\lambda_A,t)\subseteq L(\in\lambda_A,\acute{t})$,

(6) If $t,\acute{t}\in(0,1]$, $t\leq\acute{t}$. Then $L(q\lambda_A,t)\supseteq L(q\lambda_A,\acute{t})$,

(7) If $t,\acute{t}\in(0,0.5]$, $t\leq\acute{t}$. Then $L(\in\vee q\lambda_A,t)\supseteq L(\in\vee q\lambda_A,\acute{t})$,

(8) If $t,\acute{t}\in(0.5,1]$, $t\leq\acute{t}$. Then $L(\in\vee q\lambda_A,t)\subseteq L(\in\vee q\lambda_A,\acute{t})$.

Proof. The proofs are straightforward. □

Theorem 3.10. Let $\{A_i=(\mu_{A_i},\lambda_{A_i})|i\in I\}\subseteq IFS(X)$, where I is a non-empty index set. Then for all $t\in(0,1]$,

(1) $\bigcup_{i\in I}(U(\alpha\mu_{A_i},t))\subseteq U(\alpha\bigcup_{i\in I}\mu_{A_i},t)$,

(2) $\bigcap_{i\in I}(U(\alpha\mu_{A_i},t)) \supseteq U(\alpha\bigcap_{i\in I}\mu_{A_i},t)$,

(3) $\bigcup_{i\in I}(L(\alpha\lambda_{A_i},t)) = L(\alpha\bigcup_{i\in I}\lambda_{A_i},t)$,

(4) $\bigcap_{i\in I}(L(\alpha\lambda_{A_i},t)) \subseteq L(\alpha\bigcap_{i\in I}\lambda_{A_i},t)$,

where $\alpha \in \{\in, q, \in\vee q\}$.

Proof. (1) Let $x \in \bigcup_{i\in I}(U(\in\vee q\mu_{A_i},t))$. Then, there exist a $i_1 \in I$ such that $x \in U(\in\mu_{A_{i_1}},t)$ or $x \in U(q\mu_{A_{i_1}},t)$. Hence $\mu_{A_{i_1}}(x) \geq t$ or $\mu_{A_{i_1}}(x)+t > 1$ and so $sup_{i\in I}\mu_{A_i}(x) \geq t$ or $sup_{i\in I}\mu_{A_i}(x)+t > 1$. This shows that $x \in U(\in\bigcup_{i\in I}\mu_{A_i},t)\bigcup U(q\bigcup_{i\in I}\mu_{A_i},t)$ and thus $x \in U(\in\vee q\bigcup_{i\in I}\mu_{A_i},t)$.
The others are analogous.

□

Theorem 3.11. Let $A = (\mu_A, \lambda_A) \in IFS(X)$ and $\{t_i \mid i \in I\}$ be a non-empty subset of $(0,1]$. Let $t = \inf_{i\in I} t_i$ and $\acute{t} = \sup_{i\in I} t_i$. Then the following assertions hold:

(1) $\bigcup_{i\in I}(U(\in\mu_A,t_i)) = U(\in\mu_A,t)$, (2) $\bigcup_{i\in I}(U(q\mu_A,t_i)) = U(q\mu_A,\acute{t})$,

(3) $t_i \in (0,0.5] \Longrightarrow \bigcup_{i\in I}(U(\in\vee q\mu_A,t_i)) \subseteq U(\in\vee q\mu_A,t)$,

(4) $t_i \in (0.5,1] \Longrightarrow \bigcup_{i\in I}(U(\in\vee q\mu_A,t_i)) \subseteq U(\in\vee q\mu_A,\acute{t})$,

(5) $\bigcap_{i\in I}(U(\in\mu_A,t_i)) = U(\in\mu_A,\acute{t})$, (6) $\bigcap_{i\in I}(U(q\mu_A,t_i)) \subseteq U(q\mu_A,\acute{t})$,

(7) $\bigcap_{i\in I}(U(\in\vee q\mu_A,t_i)) \subseteq U(\in\vee q\mu_A,\acute{t})$,

(8) $\bigcup_{i\in I}(L(\in\mu_A,t_i)) = L(\in\mu_A,\acute{t})$, (9) $\bigcup_{i\in I}(L(q\mu_A,t_i)) = L(q\mu_A,t)$,

(10) $t_i \in (0.5,1] \Longrightarrow \bigcup_{i\in I}(L(\in\vee q\mu_A,t_i)) \subseteq L(\in\vee q\mu_A,\acute{t})$,

(11) $t_i \in (0,0.5] \Longrightarrow \bigcup_{i\in I}(L(\in\vee q\mu_A,t_i)) \subseteq L(\in\vee q\mu_A,t)$,

(12) $\bigcup_{i\in I}(L(\in\mu_A,t_i)) = L(\in\mu_A,\acute{t})$, (13) $\bigcup_{i\in I}(L(q\mu_A,t_i)) = L(q\mu_A,t)$,

(14) $\bigcap_{i\in I}(L(\in\mu_A,t_i)) = L(\in\mu_A,t)$, (15) $\bigcap_{i\in I}(L(q\mu_A,t_i)) = L(q\mu_A,\acute{t})$.

Theorem 3.12. Let $A = (\mu_A, \lambda_A) \in IFS(X)$. Then for all $t \in (0,1]$,

(1) $\mu_A = \bigcup_{t\in\mu_A(X)} t_{U(\in\mu_A,t)}$,

(2) $\lambda_A = \bigcap_{t\in\lambda_A(X)} t_{L(\in\lambda_A,t)} = \bigcap_{t\in\lambda^c_A(X)} t_{L(q\lambda_A,t)}$.

Proof. (2) Let $x \in X$. Then, we have $(\bigcap_{t\in\lambda_A(X)} t_{L(\in\lambda_A,t)})(x) = \bigcap_{t\in\lambda_A(X)} t_{L(\in\lambda_A,t)}(x)$ $= \bigwedge\{t \in \lambda_A(X)|x \in L(\in \lambda_A,t)\} = \bigwedge\{t \in \lambda_A(X)|\lambda_A(x) \leq t\} = \lambda_A(x)$.
We have $(\bigcap_{t\in\lambda^c_A(X)} t_{L(\in\lambda_A,t)})(x) = \bigcap_{t\in\lambda^c_A(X)} t_{L(\in\lambda_A,t)}(x) = \bigwedge\{1-t \in \lambda_A(X)|x \in L(q\lambda_A,t)\} = \bigwedge\{1-t \in \lambda_A(X)|\lambda_A(x)+t \leq 1\} = \lambda_A(x) = \bigwedge\{1-t \in \lambda_A(X)|\lambda_A(x) \leq 1-t\} = \lambda_A(x)$. □

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