Generating Functions of Special Triple Hypergeometric Functions

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Abstract

In this paper we have obtained some generating relations involving Appell's, Exton's, Horn's, Lauricella's and Saran's functions. We obtained these relation by using integral representation of Exton's functions $X_{11}, X_{12}, \dots, X_{20}$ in terms of Laplace's integral of several variables. Through the development of these new generating relations, we recover some Exton's, Srivastva's and Karlsson's results.

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1 Introduction

The unification of generating functions has great importance in connection with ideas and principles of special functions. In this directions some important steps has been made by researchers namely Singhal and Srivastava [13],

Chatterjea [1, 2] and Chongdar [3]. In their study Desale and Qashash [4] have obtained a new general class of generating functions for the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$ by group theoretic method. Also, they have introduced the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials with the help of two linear partial differential operators. Further, they continued their study and they used the group theoretic method to obtain the proper and improper partial bilateral as well as trilateral generating functions. In this consent one may refer their sequel of papers [5, 6].

Now continuing the work in connection with class of generating functions, we extend our ideas to obtain the new generating relations that involves between Exton's functions and hyper geometric functions, in particularly Appell's functions, Lauricella's, Horn's, Saran's and Gaussian hyper geometric functions. We used integral form of Exton's functions (in the form of Laplace integral) to obtain the new generating relations among Exton's and hypergeometric functions.

Exton in [9, 10] gave integral representation of some hyper geometric functions of three variables out of which we referred here $X_{11}, X_{12}, \dots, X_{20}$ ¹ and these relations are defined as follows (cf. [9, 10]).

$$X_{11}(a, b, c, d; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c)_{m+p}(d)_n m! n! p!}, \quad (1.1)$$

$$X_{12}(a, b, c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \quad (1.2)$$

$$X_{13}(a, b, c, d; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p x^m y^n z^p}{(d)_{m+n+p} m! n! p!}, \quad (1.3)$$

$$X_{14}(a, b, c, d, d'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p x^m y^n z^p}{(d)_{m+n}(d')_p m! n! p!}, \quad (1.4)$$

$$X_{15}(a, b, c, d, d'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p x^m y^n z^p}{(d)_m (d')_{n+p} m! n! p!}, \quad (1.5)$$

$$X_{16}(a, b, c, d, d'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p x^m y^n z^p}{(d)_{m+p}(d')_n m! n! p!}, \quad (1.6)$$

$$X_{17}(a, b, c, d_1, d_2, d_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p x^m y^n z^p}{(d_1)_m (d_2)_n (d_3)_p m! n! p!}, \quad (1.7)$$

¹In our earlier work, we determined the generating relations by integral transformation of $X_1, X_2, \dots, X_{10}$. More details about our results may be referred from [7]

$$X_{18}(a, b, b', c; d; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p x^m y^n z^p}{(d)_{m+n+p} m! n! p!}, \quad (1.8)$$

$$X_{19}(a, b, b', c; d, d'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p x^m y^n z^p}{(d)_m (d')_{n+p} m! n! p!}, \quad (1.9)$$

$$X_{20}(a, b, b', c; d, d'; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(b')_p(c)_p x^m y^n z^p}{(d)_{m+n} (d')_n m! n! p!}. \quad (1.10)$$

The integral representation of these Exton functions in terms of Laplace integral are given in the following section.

2 Integral Representation of Exton Functions

Here, we expressed Exton functions $X_{11}, X_{12}, \dots, X_{20}$ in terms of multiple integrals with the help of Laplace transform. Following are these integral representation of Exton functions.

$$X_{11}(a, b; c, d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1} \cdot {}_0F_1(-; c; xs^2 + zt^2) {}_0F_1(-; d; yst) ds dt, \quad (2.1)$$

$$X_{12}(a, b; c_1, c_2, c_3; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1} \cdot {}_0F_1(-; c_1; xs^2) {}_0F_1(-; c_2; yst) {}_0F_1(-; c_3; zs^2) ds dt, \quad (2.2)$$

$$X_{13}(a, b, c; d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty \exp(-s-t-u) s^{a-1} t^{b-1} u^{c-1} \cdot {}_0F_1(-; d; xs^2 + yst + ztu) ds dt du, \quad (2.3)$$

$$X_{14}(a, b, c; d, d'; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1} \cdot {}_0F_1(-; d; xs^2 + yst) {}_1F_1(c; d'; zt) ds dt, \quad (2.4)$$

$$X_{15}(a, b, c; d, d'; x, y, z) = \frac{1}{\Gamma(a)\Gamma(c)} \int_0^\infty \int_0^\infty \exp(-s-t) \cdot s^{a-1} t^{c-1} {}_0F_1(-; d; xs^2) {}_1F_1(b; d'; ys+zt) ds dt, \quad (2.5)$$

$$X_{16}(a, b, c; d, d'; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty \exp(-s-t-u) \cdot s^{a-1} t^{b-1} u^{c-1} {}_0F_1(-; d; xs^2 + ztu) {}_0F_1(-; d'; yst) ds dt du, \quad (2.6)$$

$$X_{17}(a, b, c; d_1, d_2, d_3; x, y, z) = \frac{1}{\Gamma(a)\Gamma(c)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{c-1} \cdot {}_0F_1(-; d_1; xs^2) \Psi_2(b; d_2, d_3; ys+zt) ds dt, \quad (2.7)$$

$$X_{18}(a, b, b', c; d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{(-s-t-u-v)} \cdot s^{a-1} t^{b-1} u^{b'-1} v^{c-1} {}_0F_1(-; d; xs^2 + yst + zuv) ds dt du dv, \quad (2.8)$$

$$X_{19}(a, b, b', c; d, d'; x, y, z) = \frac{1}{\Gamma(a)\Gamma(c)} \int_0^\infty \int_0^\infty \exp(-s-t) \cdot s^{a-1} t^{c-1} {}_0F_1(-; d; xs^2) \Phi_2(b, b'; d'; ys+zt) ds dt, \quad (2.9)$$

$$X_{20}(a, b, b', c; d, d'; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b')\Gamma(c)} \int_0^\infty \int_0^\infty \int_0^\infty e^{(-s-t-u)} s^{a-1} \cdot t^{b'-1} u^{c-1} {}_0F_1(-; d; xs^2 + ztu) {}_1F_1(b; d'; ys) ds dt du. \quad (2.10)$$

3 New Generating Relations

In this section, we used the integral representation of Exton functions to determine new generating functions which are listed in the form of following

relations (3.1) to (3.18).

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{11}(\alpha + n, \beta + n; \gamma, \delta; x, yk, z) = (1 + y)^{-\alpha} (1 + k)^{-\beta} \\
 & \cdot \sum_{n,r,m=0}^{\infty} \frac{(\alpha + n)_r (\alpha + n)_m (\delta)_{r+m}}{(\delta)_r (\delta)_m n! m! r!} \left(\frac{y}{1 + y} \right)^r \left(\frac{k}{1 + k} \right)^m \left(\frac{w}{(1 + y)(1 + k)} \right)^n \\
 & \cdot F_3 \left(\frac{\alpha + n + r}{2}, \frac{\beta + n + m}{2}, \frac{\alpha + n + r + 1}{2}, \right. \\
 & \left. \frac{\beta + n + m + 1}{2}; \gamma; \frac{4x}{(1 + y)^2}, \frac{4z}{(1 + k)^2} \right),
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{11}(\alpha + n, \beta + n; \gamma, \delta; x, yk, z) = (1 + y)^{-\alpha} (1 + k)^{-\beta} \\
 & \cdot \sum_{n,p,q=0}^{\infty} \frac{(\alpha + n)_{2p} (\beta + n)_{2q}}{(\gamma)_{p+q} n! p! q!} \left(\frac{x}{(1 + y)^2} \right)^p \left(\frac{z}{(1 + k)^2} \right)^q \left(\frac{w}{(1 + y)(1 + k)} \right)^n \\
 & \cdot F_2 \left(\delta, \alpha + n + 2p, \beta + n + 2q; \delta, \delta; \frac{y}{1 + y}, \frac{k}{1 + k} \right),
 \end{aligned} \tag{3.2}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{12}(\alpha + n, \beta + n; \gamma, \delta, \lambda; x^2, y, z^2) \\
 & = (1 + 2x)^{-\alpha} (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1 + 2x)(1 + 2z)} \right)^n \\
 & \cdot F_k \left(\gamma - \frac{1}{2}, \beta + n, \beta + n, \alpha + n, \lambda - \frac{1}{2}, \alpha + n; 2\gamma - 1, 2\lambda - 1, \delta; \right. \\
 & \left. \frac{4x}{(1 + 2x)}, \frac{4z}{(1 + 2z)}, \frac{y}{(1 + 2x)(1 + 2z)} \right),
 \end{aligned} \tag{3.3}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{12}(\alpha + n, \beta + n; \gamma, \delta, \lambda; x^2, y, z) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n \\
 & \cdot X_{17} \left(\beta + n, \alpha + n, \gamma - \frac{1}{2}; \lambda, \delta, 2\gamma - 1; z, \frac{y}{(1 + 2x)}, \frac{4x}{(1 + 2x)} \right),
 \end{aligned} \tag{3.4}$$

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{12}(\alpha + n, \beta + n; \gamma, \delta, \lambda; x, y, z^2) = (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2z} \right)^n \\
 & \cdot X_{17} \left(\alpha + n, \beta + n, \lambda - \frac{1}{2}; \gamma, \delta, 2\lambda - 1; x, \frac{y}{(1 + 2z)}, \frac{4z}{(1 + 2z)} \right),
 \end{aligned} \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{14}(\alpha + n, \beta + n, \gamma; \delta, \epsilon; x, y, z) = (1 - z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n \cdot X_{14} \left(\alpha + n, \beta + n, \epsilon - \gamma; \delta, \epsilon; x, \frac{y}{(1 - z)}, \frac{z}{(z - 1)} \right), \quad (3.6)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{14}(\alpha + n, \beta + n, \gamma; \delta, \gamma; x, y, z) = (1 - z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n \cdot H_3 \left(\alpha + n, \beta + n; \delta; x, \frac{y}{(1 - z)} \right), \quad (3.7)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{15}(\alpha + n, \beta, \gamma + n; \delta, \epsilon; x^2, y, z) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n \cdot F_M \left(\delta - \frac{1}{2}, \beta, \beta, \alpha + n, \gamma + n, \alpha + n; 2\delta - 1, \epsilon, \epsilon; \frac{4x}{(1 + 2x)}, z, \frac{y}{(1 + 2x)} \right), \quad (3.8)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{15}(\alpha + n, \beta, \gamma + n; \delta, \beta; x, y, z) = (1 - y)^{-\alpha} (1 - z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 - y)(1 - z)} \right)^n {}_2F_1 \left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \delta; \frac{4x}{(1 - y)^2} \right), \quad (3.9)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{15}(\alpha + n, \beta, \gamma + n; \delta, \beta; x^2, y, z) \\ &= (1 + 2x - y)^{-\alpha} (1 - z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1 + 2x - y)(1 - z)} \right)^n \\ & \cdot {}_2F_1 \left(\alpha + n, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{(1 + 2x - y)} \right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{17}(\alpha + n, \beta, \gamma + n; \delta_1, \delta_2, \delta_3; x^2, y, z) \\ &= (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1 + 2x)} \right)^n \\ & \cdot F_K \left(\delta_1 - \frac{1}{2}, \beta, \beta, \alpha + n, \gamma + n, \alpha + n; 2\delta_1 - 1, \delta_3, \delta_2; \frac{4x}{1 + 2x}, z, \frac{y}{1 + 2x} \right), \end{aligned} \quad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{17}(\alpha + n, \beta, \gamma + n; \delta, \beta, \beta; x, y, z) = (1-y)^{-\alpha} (1-z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1-y)(1-z)} \right)^n H_4 \left(\alpha + n, \gamma + n; \delta, \beta; \frac{x}{(1-y)^2}, \frac{yz}{(1-y)(1-z)} \right), \quad (3.12)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{17}(\alpha + n, \beta, \gamma + n; \delta, \beta, \beta; x^2, y, z) \\ &= (1+2x-y)^{-\alpha} (1-z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+2x-y)(1-z)} \right)^n \\ & F_2 \left(\alpha + n, \delta - \frac{1}{2}, \gamma + n; 2\delta - 1, \beta; \frac{4x}{(1+2x-y)}, \frac{yz}{(1+2x-y)(1-z)} \right), \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{19}(\alpha + n, \beta_1, \beta_2, \gamma + n; \delta_1, \delta_2; x^2, y, z) \\ &= (1+2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+2x)} \right)^n F_N \left(\delta_1 - \frac{1}{2}, \beta_2, \beta_1, \alpha + n, \alpha + n, \alpha + n; \right. \\ & \left. 2\delta_1 - 1, \delta_2, \delta_2; \frac{4x}{(1+2x)}, z, \frac{y}{(1+2x)} \right), \end{aligned} \quad (3.14)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{19}(\alpha + n, \beta, \delta_2 - \beta, \gamma + n; \delta_1, \delta_2; x, y, z) = (1-z)^{-\alpha} \\ & \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1-z)} \right)^n X_{15} \left(\alpha + n, \beta, \gamma + n; \delta_1, \delta_2; x, y, \frac{z}{(z-1)} \right), \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{19}(\alpha + n, \beta, \delta_2 - \beta, \gamma + n; \delta_1, \delta_2; x^2, y, z) \\ &= (1+2x)^{-\alpha} (1-z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+2x)(1-z)} \right)^n \\ & \cdot F_M \left(\delta_1 - \frac{1}{2}, \beta, \beta, \alpha + n, \alpha + n, \alpha + n; 2\delta_1 - 1, \delta_2, \delta_2; \frac{4x}{(1+2x)}, \frac{z}{(z-1)}, \frac{y}{(1+2x)} \right), \end{aligned} \quad (3.16)$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{20}(\alpha + n, \beta_1, \beta_2 + n, \gamma + n; \delta_1, \delta_2; x, y, z) \\
&= (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1-y)} \right)^n \\
& \cdot X_{20} \left(\alpha + n, \delta_2 - \beta_1, \beta_2 + n, \gamma + n; \delta_1, \delta_2; \frac{x}{(1-y)^2}, \frac{y}{(y-1)}, z \right),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{w^n}{n!} X_{20}(\alpha + n, \epsilon, \beta + n, \gamma + n; \delta, \epsilon; x, y, z) \\
&= (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1-y)} \right)^n \\
& \cdot F_3 \left(\frac{\alpha + n}{2}, \beta + n, \frac{\alpha + n + 1}{2}, \gamma + n; \delta; \frac{4x}{(1-y)^2}, z \right).
\end{aligned} \tag{3.18}$$

In above relations F_1, F_2, F_3 are the Appell functions (cf. [14], [15]), ${}_2F_1$ is called Gaussian hypergeometric function (cf. [15], pp. 13(1) and pp. 18(17)), H_3, H_4 are Horn functions (see.[15], pp. 24(6-15) and [14], pp. 33(22-31)) and F_K, F_M, F_N are Saran functions of three variables (cf. [14], pp. 67, [15])

4 Proof of Generating relations

To prove the above relations, we need the following formulae (cf. [14], [15], [8] and [12]).

$$\sum_{m,n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!}, \tag{4.1}$$

$$\Psi_2(c; c, c; x, y) = \exp(x+y) {}_0F_1(-; c; xy), \tag{4.2}$$

$$\Phi_2(a, c-a; c; x, y) = \exp(y) {}_1F_1(a; c; x-y), \tag{4.3}$$

$${}_1F_1(a; c; x) = \exp(x) {}_1F_1(c-a; c; -x), \tag{4.4}$$

$${}_0F_1(-; c; x^2) = \exp(-2x) {}_1F_1(c - \frac{1}{2}; 2c-1; 4x), \tag{4.5}$$

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, \dots, \tag{4.6}$$

$$(\lambda)_{m+n} = (\lambda)_m(\lambda + m)_n, \quad (4.7)$$

$$(\lambda)_{2n} = 2^{2n} \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n, \quad n = 0, 1, 2, \dots, \quad (4.8)$$

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}, \quad 0 \leq k \leq n, \quad (4.9)$$

$$L\{t^{2\sigma-1} {}_mF_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; \lambda^2 t^2)\} = \Gamma(2\sigma) p^{-2\sigma} {}_{m+2}F_n\left(\alpha_1, \dots, \alpha_n, \frac{\sigma}{2}, \frac{\sigma+1}{2}; \beta_1, \dots, \beta_n; 4\lambda^2 p^{-2}\right) \quad (4.10)$$

$$L\{t^{\gamma-1} {}_1F_1(\alpha, \gamma; \lambda t)\} = \Gamma(\gamma) p^{\alpha-\gamma} (p-\lambda)^{-\alpha}, \\ \operatorname{Re}(\gamma) > 0, \operatorname{Re}(p) > 0, \operatorname{Re}(\lambda) > 0. \quad (4.11)$$

$$L\{t^v\} = \Gamma(v+1) p^{-v-1}, \quad \operatorname{Re}(v) > -1, \operatorname{Re}(p) > 0. \quad (4.12)$$

$$L\{x^\mu {}_1F_1(a_1; b_1; \sigma x) {}_1F_1(a; b; \omega x)\} \\ = \Gamma(\mu+1) p^{-\mu-1} F_2\left(\mu+1, a, a_1; b, b_1; \frac{\omega}{p}, \frac{\sigma}{p}\right), \quad (4.13) \\ \operatorname{Re}(\mu) > -1, \operatorname{Re}(p-\sigma), \operatorname{Re}(p-\omega), \operatorname{Re}(p-\sigma-\omega) > 0.$$

Where L is the Laplace transform, Φ_2, Ψ_2 are Humbert's functions, ${}_1F_1$ is Kummer's function (or confluent hypergeometric function) and ${}_mF_n$ is Generalized hypergeometric function.

Proof of relation 3.1:

Let us denote the left hand side of (3.1) by A , using (2.1)

$$A = \sum_{n=0}^{\infty} \frac{w^n}{n! \Gamma(\alpha+n) \Gamma(\beta+n)} \int_0^{\infty} \int_0^{\infty} \exp(-s-t) s^{\alpha+n-1} t^{\beta+n-1} \\ \cdot {}_0F_1(-; \gamma; xs^2 + zt^2) {}_0F_1(-; \delta; ykst) ds dt$$

by using (4.1) and (4.2), we get

$$A = \sum_{n,p,q=0}^{\infty} \frac{w^n x^p z^q}{(\gamma)_{p+q} n! p! q! \Gamma(\alpha+n) \Gamma(\beta+n)} \int_0^{\infty} \int_0^{\infty} e^{(-s(1+y))} e^{(-t(1+k))} \\ \cdot s^{\alpha+n+2p-1} t^{\beta+n+2q-1} \Psi_2(\delta; \delta, \delta; ys, kt) ds dt$$

The function Ψ_2 which appears in above equation can be replaced by its series form and then interchanging the order of the summation and integral sign which is permissible here, we get

$$A = \sum_{n,p,q,r,m=0}^{\infty} \frac{(\delta)_{r+m} w^n x^p z^q y^r k^m}{(\gamma)_{p+q} (\delta)_r (\delta)_m n! p! q! r! m! \Gamma(\alpha+n) \Gamma(\beta+n)} \int_0^{\infty} \int_0^{\infty} e^{(-s(1+y))} \cdot e^{(-t(1+k))} s^{\alpha+n+2p+r-1} t^{\beta+n+2q+m-1} ds dt$$

Now, we use sequentially (4.12), (4.6), (4.7) and (4.8); and then simplified by using series manipulation, which complete the proof of relation (3.1).

Remark 1. *The proof of all remaining relations runs in the same way, considering the appropriate integral representation and Laplace transform during the proof.*

5 Special Cases

Some generating relations, which are belived to be new, can be established in this section as the special cases of the results were obtained in previous section.

1. If we put $z = 0$ in (3.1), then we have generating relation between Horn's function H_4 and Exton's function X_{17} .

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha+n, \beta+n; \gamma, \delta; x, yk) = (1+y)^{-\alpha} (1+k)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1+y)(1+k)} \right)^n X_{17} \left(\alpha+n, \delta, \beta+n; \gamma, \delta, \delta; \frac{x}{(1+y)^2}, \frac{y}{1+y}, \frac{k}{1+k} \right) \quad (5.1)$$

2. Now assign the value to x and z to be zero in (3.2), we get the relation between Gaussian hypergeometric function and Appell's function.

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1(\alpha+n, \beta+n; \delta; yk) = (1+y)^{-\alpha} (1+k)^{-\beta} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+y)(1+k)} \right)^n F_2 \left(\delta, \alpha+n, \beta+n; \delta, \delta; \frac{y}{1+y}, \frac{k}{1+k} \right). \quad (5.2)$$

3. Put $x = 0$ in equations (3.3), (3.5), (3.6), (3.7), (3.12), and (3.15), we

obtained the following generating relations.

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\beta + n, \alpha + n; \lambda, \delta; z^2, y) = (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1 + 2z)} \right)^n \cdot F_2 \left(\alpha + n, \lambda - \frac{1}{2}, \alpha + n; 2\lambda - 1, \delta; \frac{4z}{(1 + 2z)}, \frac{y}{(1 + 2z)} \right) \quad (5.3)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\beta + n, \alpha + n; \lambda, \delta; z^2, y) = (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2z} \right)^n \cdot F_2 \left(\beta + n, \alpha + n, \lambda - \frac{1}{2}; \delta, 2\lambda - 1; \frac{y}{(1 + 2z)}, \frac{4z}{(1 + 2z)} \right) \quad (5.4)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\beta + n, \alpha + n, \gamma; \delta, \epsilon; y, z) = (1 - z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{1 - z} \right)^n F_2 \left(\beta + n, \alpha + n, \epsilon - \gamma; \delta, \epsilon; \frac{y}{(1 - z)}, \frac{z}{(z - 1)} \right) \quad (5.5)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\beta + n, \alpha + n, \gamma; \delta, \gamma; y, z) = (1 - z)^{-\beta} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n {}_2F_1 \left(\alpha + n, \beta + n; \delta; \frac{y}{(1 - z)} \right) \quad (5.6)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\beta, \alpha + n, \gamma + n; \beta, \beta; y, z) = (1 - y)^{-\alpha} (1 - z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 - y)(1 - z)} \right)^n {}_2F_1 \left(\alpha + n, \gamma + n; \beta; \frac{yz}{(1 - y)(1 - z)} \right) \quad (5.7)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_3(\alpha + n, \delta_2 - \beta, \beta, \gamma + n; \delta_2; y, z) = (1 - z)^{-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 - z)} \right)^n F_1 \left(\beta, \alpha + n, \gamma + n; \delta_2; y, \frac{z}{(z - 1)} \right) \quad (5.8)$$

4. Now substituting $z = 0$ in (3.4), (3.8), (3.9), (3.10), (3.11), (3.14), (3.17), and (3.18), we obtain the following relations.

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta + n; \gamma, \delta; x^2, y) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n \cdot F_2 \left(\alpha + n, \beta + n, \gamma - \frac{1}{2}; \delta, 2\gamma - 1; \frac{y}{(1 + 2x)}, \frac{4x}{(1 + 2x)} \right) \quad (5.9)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \epsilon; x^2, y) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n \cdot F_2 \left(\alpha + n, \beta, \delta - \frac{1}{2}; \epsilon, 2\delta - 1; \frac{y}{(1 + 2x)}, \frac{4x}{(1 + 2x)} \right) \quad (5.10)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \beta; x, y) = (1 - y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 - y)} \right)^n {}_2F_1 \left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \delta; \frac{4x}{(1 - y)^2} \right) \quad (5.11)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \beta; x^2, y) = (1 + 2x - y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 + 2x - y)} \right)^n {}_2F_1 \left(\alpha + n, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{(1 + 2x - y)} \right) \quad (5.12)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta_1, \delta_2; x^2, y) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 + 2x)} \right)^n F_2 \left(\alpha + n, \delta_1 - \frac{1}{2}, \beta; 2\delta_1 - 1, \delta_2; \frac{4x}{1 + 2x}, \frac{y}{1 + 2x} \right) \quad (5.13)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta_1; \delta_1, \delta_2; x^2, y) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1 + 2x)} \right)^n F_2 \left(\alpha + n, \delta_1 - \frac{1}{2}, \beta_1; 2\delta_1 - 1, \delta_2; \frac{4x}{(1 + 2x)}, \frac{y}{(1 + 2x)} \right) \quad (5.14)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta_1; \delta_1, \delta_2; x, y) = (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1-y)} \right)^n H_4 \left(\alpha + n, \delta_2 - \beta_1; \delta_1, \delta_2; \frac{x}{(1-y)^2}, \frac{y}{(y-1)} \right) \quad (5.15)$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \epsilon; \delta, \epsilon; x, y) = (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1-y)} \right)^n {}_2F_1 \left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \delta; \frac{4x}{(1-y)^2} \right) \quad (5.16)$$

5. Here, we determine some new relations, which are obtained by substituting $n = 0$ into the equations (3.1), (3.2), (3.10), (3.13), and (3.16).

$$X_{11}(\alpha, \beta; \gamma, \delta; x, yk, z) = (1+y)^{-\alpha}(1+k)^{-\beta} \sum_{r,m=0}^{\infty} \frac{(\alpha)_r(\alpha)_m(\delta)_{r+m}}{(\delta)_r(\delta)_m m! r!} \cdot \left(\frac{y}{1+y} \right)^r \left(\frac{k}{1+k} \right)^m F_3 \left(\frac{\alpha+r}{2}, \frac{\beta+m}{2}, \frac{\alpha+r+1}{2}, \frac{\beta+m+1}{2}; \gamma; \frac{4x}{(1+y)^2}, \frac{4z}{(1+k)^2} \right) \quad (5.17)$$

$$X_{11}(\alpha, \beta; \gamma, \delta; x, yk, z) = (1+y)^{-\alpha}(1+k)^{-\beta} \sum_{p,q=0}^{\infty} \frac{(\alpha)_{2p}(\beta)_{2q}}{(\gamma)_{p+q} p! q!} \cdot \left(\frac{x}{(1+y)^2} \right)^p \left(\frac{z}{(1+k)^2} \right)^q F_2 \left(\delta, \alpha + 2p, \beta + 2q; \delta, \delta; \frac{y}{1+y}, \frac{k}{1+k} \right) \quad (5.18)$$

$$X_{15}(\alpha, \beta, \gamma; \delta, \beta; x^2, y, z) = (1+2x-y)^{-\alpha}(1-z)^{-\gamma} \cdot {}_2F_1 \left(\alpha, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{(1+2x-y)} \right) \quad (5.19)$$

$$X_{17}(\alpha, \beta, \gamma; \delta, \beta, \beta; x^2, y, z) = (1+2x-y)^{-\alpha}(1-z)^{-\gamma} \cdot F_2 \left(\alpha, \delta - \frac{1}{2}, \gamma; 2\delta - 1, \beta; \frac{4x}{(1+2x-y)}, \frac{yz}{(1+2x-y)(1-z)} \right) \quad (5.20)$$

$$X_{19}(\alpha, \beta, \delta_2 - \beta, \gamma; \delta_1, \delta_2; x^2, y, z) = (1+2x)^{-\alpha}(1-z)^{-\gamma} \cdot F_M \left(\delta_1 - \frac{1}{2}, \beta, \beta, \alpha, \alpha, \alpha; 2\delta_1 - 1, \delta_2, \delta_2; \frac{4x}{(1+2x)}, \frac{z}{(z-1)}, \frac{y}{(1+2x)} \right) \quad (5.21)$$

6. The following results are the Exton's results, which are obtained by Exton in his paper (cf. [10]). We recover these result by assigning the value zero

to n into the equations (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18), we get

$$\begin{aligned} X_{12}(\alpha, \beta; \gamma, \delta, \lambda; x^2, y, z^2) &= (1+2x)^{-\alpha}(1+2z)^{-\beta} \\ &\cdot F_k \left(\gamma - \frac{1}{2}, \beta, \beta, \alpha, \lambda - \frac{1}{2}, \alpha; 2\gamma - 1, 2\lambda - 1, \delta; \right. \\ &\quad \left. \frac{4x}{(1+2x)}, \frac{4z}{(1+2z)}, \frac{y}{(1+2x)(1+2z)} \right) \end{aligned} \quad (5.22)$$

$$\begin{aligned} X_{12}(\alpha, \beta; \gamma, \delta, \lambda; x^2, y, z) &= (1+2x)^{-\alpha} \\ &\cdot X_{17} \left(\beta, \alpha, \gamma - \frac{1}{2}; \lambda, \delta, 2\gamma - 1; z, \frac{y}{(1+2x)}, \frac{4x}{(1+2x)} \right) \end{aligned} \quad (5.23)$$

$$\begin{aligned} X_{12}(\alpha, \beta; \gamma, \delta, \lambda; x, y, z^2) &= (1+2z)^{-\beta} \\ &\cdot X_{17} \left(\alpha, \beta, \lambda - \frac{1}{2}; \gamma, \delta, 2\lambda - 1; x, \frac{y}{(1+2z)}, \frac{4z}{(1+2z)} \right) \end{aligned} \quad (5.24)$$

$$\begin{aligned} X_{14}(\alpha, \beta, \gamma; \delta, \epsilon; x, y, z) \\ = (1-z)^{-\beta} X_{14} \left(\alpha, \beta, \epsilon - \gamma; \delta, \epsilon; x, \frac{y}{(1-z)}, \frac{z}{(z-1)} \right) \end{aligned} \quad (5.25)$$

$$X_{14}(\alpha, \beta, \gamma; \delta, \gamma; x, y, z) = (1-z)^{-\beta} H_3 \left(\alpha, \beta; \delta; x, \frac{y}{(1-z)} \right) \quad (5.26)$$

$$\begin{aligned} X_{15}(\alpha, \beta, \gamma; \delta, \epsilon; x^2, y, z) &= (1+2x)^{-\alpha} \\ &\cdot F_M \left(\delta - \frac{1}{2}, \beta, \beta, \alpha, \gamma, \alpha; 2\delta - 1, \epsilon, \epsilon; \frac{4x}{(1+2x)}, z, \frac{y}{(1+2x)} \right) \end{aligned} \quad (5.27)$$

$$\begin{aligned} X_{15}(\alpha, \beta, \gamma; \delta, \beta; x, y, z) \\ = (1-y)^{-\alpha}(1-z)^{-\gamma} {}_2F_1 \left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \delta; \frac{4x}{(1-y)^2} \right) \end{aligned} \quad (5.28)$$

$$\begin{aligned} X_{17}(\alpha, \beta, \gamma; \delta_1, \delta_2, \delta_3; x^2, y, z) &= (1+2x)^{-\alpha} \\ &\cdot F_K \left(\delta_1 - \frac{1}{2}, \beta, \beta, \alpha, \gamma, \alpha; 2\delta_1 - 1, \delta_3, \delta_2; \frac{4x}{1+2x}, z, \frac{y}{1+2x} \right) \end{aligned} \quad (5.29)$$

$$\begin{aligned} X_{17}(\alpha, \beta, \gamma; \delta, \beta, \beta; x, y, z) \\ = (1-y)^{-\alpha}(1-z)^{-\gamma} H_4 \left(\alpha, \gamma; \delta, \beta; \frac{x}{(1-y)^2}, \frac{yz}{(1-y)(1-z)} \right) \end{aligned} \quad (5.30)$$

$$X_{19}(\alpha, \beta_1, \beta_2, \gamma; \delta_1, \delta_2; x^2, y, z) = (1 + 2x)^{-\alpha} \cdot F_N \left(\delta_1 - \frac{1}{2}, \beta_2, \beta_1, \alpha, \alpha, \alpha; 2\delta_1 - 1, \delta_2, \delta_2; \frac{4x}{(1 + 2x)}, z, \frac{y}{(1 + 2x)} \right) \quad (5.31)$$

$$X_{19}(\alpha, \beta, \delta_2 - \beta, \gamma; \delta_1, \delta_2; x, y, z) = (1 - z)^{-\alpha} X_{15} \left(\alpha, \beta, \gamma; \delta_1, \delta_2; x, y, \frac{z}{(z - 1)} \right) \quad (5.32)$$

$$X_{20}(\alpha, \beta_1, \beta_2, \gamma; \delta_1, \delta_2; x, y, z) = (1 - y)^{-\alpha} \cdot X_{20} \left(\alpha, \delta_2 - \beta_1, \beta_2, \gamma; \delta_1, \delta_2; \frac{x}{(1 - y)^2}, \frac{y}{(y - 1)}, z \right) \quad (5.33)$$

$$X_{20}(\alpha, \epsilon, \beta, \gamma; \delta, \epsilon; x, y, z) = (1 - y)^{-\alpha} F_3 \left(\frac{\alpha}{2}, \beta, \frac{\alpha + 1}{2}, \gamma; \delta; \frac{4x}{(1 - y)^2}, z \right) \quad (5.34)$$

7. Following relations are the results of Srivastava's and Karlsson's. Redears may concent their book [15] for more details of these results. We recover these result by assignning the value zero to n into the equations (5.2), (5.7) and (5.8).

$${}_2F_1(\alpha, \beta; \delta; yk) = (1 + y)^{-\alpha} (1 + k)^{-\beta} F_2 \left(\delta, \alpha, \beta; \delta, \delta; \frac{y}{1 + y}, \frac{k}{1 + k} \right) \quad (5.35)$$

When we put $y = \frac{k}{1+k}$ and $x = \frac{y}{1+y}$.

$$F_2(\beta, \alpha, \gamma; \beta, \beta; y, z) = (1 - y)^{-\alpha} (1 - z)^{-\gamma} {}_2F_1 \left(\alpha, \gamma; \beta; \frac{yz}{(1 - y)(1 - z)} \right) \quad (5.36)$$

$$F_3(\alpha, \delta_2 - \beta, \beta, \gamma; \delta_2; y, z) = (1 - z)^{-\gamma} F_1 \left(\beta, \alpha, \gamma; \delta_2; y, \frac{z}{(z - 1)} \right) \quad (5.37)$$

6 Conclusion

We used the Laplace integral representation of Exton's functions given by (1.1) to (1.10) to determine the new generating relations (3.1) to (3.18). Many of these results are the relations between Appell's functions F_1, F_2, F_3 ; Horn's functions H_3, H_4 ; Saran's functions F_K, F_M, F_N and Gaussian hypergeometric function ${}_2F_1$. One of the special case of (5.1) is the generating relation between Horn's and Exton's functions. The generating relation (5.2) is the relation between Gaussian hypergeometric function and Appell's functions. Also, we have seen in the section of special cases many of the results from section 3 reduces to Exton's functions for $n = 0$.

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