A Generalization of the Inclusion-Exclusion Principle

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Abstract

In this note we present a generalization of the inclusion-exclusion principle and an application of this more general principle.

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1 Generalization of the Inclusion-Exclusion Principle

The following theorem is sometimes called either the principle of cross-classification or the inclusion-exclusion principle. We now enunciate the principle.

Theorem 1.1 (*Inclusion-exclusion principle*) Let S be a set of N distinct elements, and let $S_1, \dots, S_r$ be arbitrary subsets of S containing $N_1, \dots, N_r$ elements, respectively. For $1 \leq i < j < \dots < l \leq r$, let $S_{ij\dots l}$ be the intersection of $S_i, S_j, \dots, S_l$ and let $N_{ij\dots l}$ be the number of elements of $S_{ij\dots l}$. Then the number K of elements of S not in any of $S_1, \dots, S_r$ is

$$K = N - \sum_{1 \leq i \leq r} N_i + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \dots + (-1)^r N_{12\dots r} \quad (1)$$

Proof. See, for example, [3] (page 84) or [2] (page 233).

Theorem 1.2 *Let S be a set of N distinct elements, that is*

$$S = \{a_1, a_2, \dots, a_N\},$$

and let $S_1, \dots, S_r$ be arbitrary subsets of S containing $N_1, \dots, N_r$ elements, respectively. For $1 \leq i < j < \dots < l \leq r$, let $S_{ij\dots l}$ be the intersection of $S_i, S_j, \dots, S_l$ and let $N_{ij\dots l}$ be the number of elements of $S_{ij\dots l}$. Let f be a function with domain S which assigns to each element $a_m \in S$ ($m = 1, 2, \dots, N$) a complex number $f(a_m)$ ($m = 1, 2, \dots, N$). Finally, let $T = S - \bigcup_{i=1}^r S_i$. The following formula holds.

$$\begin{aligned} \sum_{a_m \in T} f(a_m) &= \sum_{a_m \in S} f(a_m) - \sum_{1 \leq i \leq r} \sum_{a_m \in S_i} f(a_m) + \sum_{1 \leq i < j \leq r} \sum_{a_m \in S_{i,j}} f(a_m) \\ &- \sum_{1 \leq i < j < k \leq r} \sum_{a_m \in S_{i,j,k}} f(a_m) + \dots + (-1)^r \sum_{a_m \in S_{1,2,\dots,r}} f(a_m) \end{aligned} \quad (2)$$

Proof. The proof is similar as Theorem 1.1. See, for example, [3] (page 84) or [2] (page 233).

Remark 1.3 *Note that if the function f is the constant function $f(a_m) = 1$ ($m = 1, 2, \dots, N$) then equation (2) becomes equation (1). That is, the inclusion-exclusion principle is a particular case of this more general principle, since*

$$\begin{aligned} \sum_{a_m \in T} f(a_m) &= \sum_{a_m \in T} 1 = K \\ \sum_{a_m \in S} f(a_m) &= \sum_{a_m \in S} 1 = N \\ \sum_{a_m \in S_i} f(a_m) &= \sum_{a_m \in S_i} 1 = N_i \end{aligned}$$

etc

2 An Application

If $n \geq 1$ the Euler's function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n . The following formula is well-known

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad (3)$$

where, the product run on the different prime divisors p of n . In this note p denotes a positive prime.

Let us consider the sum $1^s + 2^s + \cdots + n^s$ where s is an arbitrary but fixed positive integer. It is well known there exists a polynomial in n of degree $s + 1$ and rational coefficients such that

$$1^s + 2^s + \cdots + n^s = c_{s+1}n^{s+1} + c_s n^s + c_{s-1}n^{s-1} + c_{s-2}n^{s-2} + \cdots + c_1 n \quad (4)$$

For example, we have

$$1 + 2 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n \quad (5)$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \quad (6)$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad (7)$$

There exist various methods in the literature to obtain these polynomials.

Theorem 2.1 *Let $\varphi_s(n)$ denote the sum of the s th powers of the numbers $\leq n$ and relatively prime to n . The following formula holds (see (3) and (4))*

$$\begin{aligned} \varphi_s(n) &= c_{s+1}n^s \varphi(n) + \prod_{p|n} (1-p) c_{s-1}n^{s-1} + \prod_{p|n} (1-p^2) c_{s-2}n^{s-2} \\ &+ \cdots + \prod_{p|n} (1-p^{s-1}) c_1 n \end{aligned} \quad (8)$$

Proof. We have (see (4)) (The p_i are the different prime divisors of n)

$$\begin{aligned} (p_i 1)^s + (p_i 2)^s + \cdots + \left(p_i \frac{n}{p_i}\right)^s &= p_i^s \left(1^s + 2^s + \cdots + \left(\frac{n}{p_i}\right)^s\right) \\ &= c_{s+1} \frac{n^{s+1}}{p_i} + c_s n^s + p_i c_{s-1} n^{s-1} + p_i^2 c_{s-2} n^{s-2} + \cdots + p_i^{s-1} c_1 n \end{aligned} \quad (9)$$

$$\begin{aligned} (p_i p_j 1)^s + (p_i p_j 2)^s + \cdots + \left(p_i p_j \frac{n}{p_i p_j}\right)^s &= p_i^s p_j^s \left(1^s + 2^s + \cdots + \left(\frac{n}{p_i p_j}\right)^s\right) \\ &= c_{s+1} \frac{n^{s+1}}{p_i p_j} + c_s n^s + p_i p_j c_{s-1} n^{s-1} + p_i^2 p_j^2 c_{s-2} n^{s-2} + \cdots + p_i^{s-1} p_j^{s-1} c_1 n \end{aligned} \quad (10)$$

etc

Substituting (4), (9), (10), etc into (2) we obtain (8). The theorem is proved.

Using formulae (5), (6) and (7) we obtain the following examples of Theorem 2.1.

$$\varphi_1(n) = \frac{1}{2}n\varphi(n)$$

$$\varphi_2(n) = \frac{1}{3}n^2\varphi(n) + \prod_{p|n}(1-p)\frac{1}{6}n$$

$$\varphi_3(n) = \frac{1}{4}n^3\varphi(n) + \prod_{p|n}(1-p)\frac{1}{4}n^2$$

Other proof of Theorem 2.1 using a different method is given in [1, chapter 2].

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