Stability and Convergence of a Finite Difference Scheme for Fractional Partial Differential Equations by Matrix Method

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Abstract

Here, we searched stability and convergence of a difference scheme which is constructed for solving a fractional partial differential equation with Caputo fractional derivative. Stability is proved by matrix method. Numerical experiments are presented.

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Keywords: Fractional partial differential equations, Finite difference scheme, Matrix stability

1 Introduction

There are two fundamental types of fractional heat equations, time fractional heat equations and space fractional heat equations. Some difference schemes for the space-fractional heat equations are presented in [6][15][16]. The Crank-Nicholson method was applied directly to obtain a numerical solution for time fractional advection dispersion equations with Riemann Liouville derivative in [8].

In this work, we use a numerical approximation based on the Crank-Nicholson method for fractional derivatives. Then, the matrix stability of the method is proved conditionally. Here, we consider the following time fractional heat equation;

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), (0 < x < 1, 0 < t < 1), \\ u(x,0) = r(x), 0 \leq x \leq 1, \\ u(0,t) = 0, u(1,t) = 0, 0 \leq t \leq 1. \end{cases} \quad (1)$$

Here, the term $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes α -order Caputo derivative, with the formula:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(x,\tau)}{(t-\tau)^\alpha} d\tau, \text{ where } 0 < \alpha < 1, \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function.

2 Finite Difference Approximation to Derivatives

In this section, we introduce the basic ideas for the numerical solution of the time fractional heat equation (1) by Crank-Nicholson difference scheme.

For some positive integers M and N , the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0, 1]$ are the numbers $x_j = jh$, $j = 0, 1, 2, \dots, M$, and the grid points in the time interval $[0, 1]$ are labeled $t_k = k\tau$, $k = 0, 1, 2, \dots, N$. The values of the functions u and f at the grid points are denoted $u_j^k = u(x_j, t_k)$ and $f_j^k = f(x_j, t_k)$, respectively. Let $u(x, t)$, $u_t(x, t)$ and $u_{tt}(x, t)$ are continuous on $[0, 1]$.

Setting $\sigma = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau^\alpha}$ and $w_j = \sigma ((j+1/2)^{1-\alpha} - (j-1/2)^{1-\alpha})$, we have the following approximation[7];

$$\begin{aligned} \frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} &= \left[w_1 u^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u^m - w_k u^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] \\ &+ O(\tau^{2-\alpha}). \end{aligned} \quad (3)$$

In addition for $k = 0$ there is no these terms $w_1 u^k$ and $w_k u^0$.

On the other hand, we have

$$\frac{\partial^2 u(x_j, t_{k+\frac{1}{2}})}{\partial x^2} = \frac{1}{2} \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} \right] + O(h^2). \quad (4)$$

3 The Difference Scheme

Using the equations (3) and (4), we obtain the following difference scheme [7] which is accurate of order $O(\tau^{2-\alpha} + h^2)$;

$$\left\{ \begin{array}{l} \left[\left(-\frac{1}{2h^2} \right) u_{j+1}^{k+1} + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \right) u_j^{k+1} + \left(-\frac{1}{2h^2} \right) u_{j-1}^{k+1} \right] \\ + \left[\left(-\frac{1}{2h^2} \right) u_{j+1}^k + \left(-\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \right) u_j^k + \left(-\frac{1}{2h^2} \right) u_{j-1}^k \right] \\ + \left[w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 \right] \\ = f(x_j, t_k + \frac{\tau}{2}), \quad 0 \leq k \leq N-1, \quad 1 \leq j \leq M, \\ u_j^0 = r(x_j), \quad 1 \leq j \leq M, \\ u_0^k = 0, \quad u_M^k = 0, \quad 0 \leq k \leq N. \end{array} \right. \quad (5)$$

4 Matrix Stability Of The Difference Scheme

The difference scheme above (5) can be written in matrix form,

$$\left\{ \begin{array}{l} AU^1 = BU^0 + \varphi^0, \quad k = 0 \\ AU^{k+1} = BU^k - w_1 U^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) U^m \\ + w_k U^0 + \varphi^k, \quad 1 \leq k \leq N-1 \\ U_j^0 = r(x_j), \quad 1 \leq j \leq M-1, \\ U_0^k = 0, \quad U_M^k = 0, \quad 0 \leq k \leq N. \end{array} \right. \quad (6)$$

where $\varphi^k = [\varphi_0^k, \varphi_1^k, \varphi_2^k, \dots, \varphi_M^k]^T$, $\varphi_j^0 = r(x_j)$, $\varphi_j^k = f(x_j, t_{k+1/2})$, $1 \leq k \leq N$, $1 \leq j \leq M$, and $U^k = [U_0^k, U_1^k, U_2^k, \dots, U_M^k]^T$. Here, A and B are 3-diagonal matrices of the form :

$$A = \begin{bmatrix} \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & \frac{-1}{2h^2} & & & & \\ & \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & \frac{-1}{2h^2} \\ & & & & \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \end{bmatrix}_{(M-1) \times (M-1)}$$

$$B = \begin{bmatrix} \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} & & & & \\ & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} \\ & & & & \frac{1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} \end{bmatrix}_{(M-1) \times (M-1)}.$$

We denote $\|A\| = \|A\|_\infty = \max_{1 \leq j \leq M-1} \left\{ \sum_{m=1}^{M-1} |a_{jm}| \right\}$, where $A = [a_{jm}]_{(M-1) \times (M-1)}$.

Lemma 4.1 If $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} - w_1 > 0$ then $\|A^{-1}B\| \leq 1$.

Proof. Assume $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} - w_1 > 0$. Since $w_1 > 0$ it implies that $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} > 0$. Therefore,

$$\begin{aligned} \|A^{-1}B\| &\leq \|A^{-1}\| \|B\| \leq \frac{1}{\min_{1 \leq j \leq M-1} \left\{ |a_{jj}| - \sum_{m \neq j, m=1}^{M-1} |a_{jm}| \right\}} \|B\| \\ &\leq \frac{\left| \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} \right| + \left\{ \left| \frac{1}{2h^2} \right| + \left| \frac{1}{2h^2} \right| \right\}}{\left| \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \right| - \left\{ \left| \frac{1}{-2h^2} \right| + \left| \frac{1}{-2h^2} \right| \right\}} \\ &= \frac{\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} + \frac{1}{2h^2} + \frac{1}{2h^2}}{\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} - \frac{1}{2h^2} - \frac{1}{2h^2}} = \frac{\frac{\sigma}{2^{1-\alpha}}}{\frac{\sigma}{2^{1-\alpha}}} \\ &= 1. \end{aligned}$$

Lemma 4.2 If $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} - w_1 > 0$ then $\|A^{-1}(B - w_1 I)\| + \|A^{-1}w_1\| \leq 1$

Proof.

$$\begin{aligned} \|A^{-1}(B - w_1 I)\| + \|w_1 A^{-1}\| &\leq \|A^{-1}\| \|B - w_1 I\| + \|w_1 A^{-1}\| \\ &\leq \frac{\frac{\sigma}{2^{1-\alpha}} - w_1}{\frac{\sigma}{2^{1-\alpha}}} + \frac{w_1}{\frac{\sigma}{2^{1-\alpha}}} \\ &= 1 \end{aligned}$$

Theorem 4.1 The difference scheme (6) is stable.

Proof To prove the conditional stability of (6), let U_j^k and V_j^k be the exact and approximate solution of (6) with initial value U_j^0 and V_j^0 respectively. We denote the corresponding error by $\varepsilon_j^k = U_j^k - V_j^k$ and $\varepsilon^k = [\varepsilon_1^k, \varepsilon_2^k, \dots, \varepsilon_{M-1}^k]^t$ where $(0 \leq j \leq M, 0 \leq k \leq N)$. Then ε^k satisfies if $k = 0$

$$A\varepsilon^1 = B\varepsilon^0$$

if $k > 0$

$$A\varepsilon^{k+1} = B\varepsilon^k - w_1\varepsilon^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1})\varepsilon^m + w_k\varepsilon^0.$$

Let us prove $\|\varepsilon^k\| \leq \|\varepsilon^0\|$, $k = 0, 1, 2, \dots$ by induction. In fact, if $k = 0$

$$\varepsilon^1 = A^{-1}B\varepsilon^0$$

from that

$$\|\varepsilon^1\| = \|A^{-1}B\varepsilon^0\| \leq \|A^{-1}B\| \|\varepsilon^0\|.$$

Since $\|A^{-1}B\| \leq 1$, from the Lemma 1, we have $\|\varepsilon^1\| \leq \|\varepsilon^0\|$. If $k = 1$, then we have

$$\varepsilon^2 = A^{-1}(B - w_1 I)\varepsilon^1 + w_1 A^{-1}\varepsilon^0.$$

From the last equation, we obtain

$$\begin{aligned}
\|\varepsilon^2\| &= \|A^{-1}(B - w_1 I)\varepsilon^1 + w_1 A^{-1}\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^1\| + \|w_1 A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^0\| + \|w_1 A^{-1}\| \|\varepsilon^0\| \\
&\leq \{\|A^{-1}(B - w_1 I)\| + \|w_1 A^{-1}\|\} \|\varepsilon^0\|.
\end{aligned}$$

If the condition above is satisfied, then $\|\varepsilon^2\| \leq \|\varepsilon^0\|$ is obtained. Now, assume $\|\varepsilon^s\| \leq \|\varepsilon^0\|$ for all $s \leq k$, we will prove it is also true for $s = k + 1$.

$$\begin{aligned}
\|\varepsilon^{k+1}\| &= \left\| A^{-1} \left(B\varepsilon^k - w_1 \varepsilon^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \varepsilon^m + w_k \varepsilon^0 \right) \right\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^k\| + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \|A^{-1}\| \|\varepsilon^m\| \\
&\quad + w_k \|A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^0\| + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) \|A^{-1}\| \|\varepsilon^0\| \\
&\quad + w_k \|A^{-1}\| \|\varepsilon^0\| \\
&\leq \|A^{-1}(B - w_1 I)\| \|\varepsilon^0\| + \{(w_{k-1} - w_k) + \dots + (w_1 - w_2) + w_k\} \|A^{-1}\| \|\varepsilon^0\| \\
&\leq (\|A^{-1}(B - w_1 I)\| + w_1 \|A^{-1}\|) \|\varepsilon^0\| \\
&\leq \|\varepsilon^0\|.
\end{aligned}$$

Therefore, under the condition $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} - w_1 > 0$, the stability inequality is obtained.

5 Convergence Of The Difference Scheme

Theorem 5.1. The proposed scheme is convergent and the following estimate holds:

$$\|e^k\| \leq Z(\alpha)(\tau^{2-\alpha} + h^2),$$

with the condition $\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} - w_1 > 0$. Here $Z(\alpha)$ does not depend on τ and h .

Proof. Let $R = [c(\tau^{2-\alpha} + h^2), \dots, c(\tau^{2-\alpha} + h^2)]_{M-1}^T$ set $w_0^{-1} = 1$. Since $e_j^k = u_j^k - U_j^k$, notice that $e^0 = 0$. Firstly, we prove that $\|e^{k+1}\| \leq w_n^{-1} \|R\|$ for all n by induction. We have the following error equation when $k = 0$.

$$e^1 = A^{-1}Be^0 + A^{-1}R = A^{-1}R$$

$$\|e^1\| = \|A^{-1}R\| \leq \|A^{-1}\| \|R\| \leq w_0^{-1} \|R\|.$$

Similarly, for $k = 2$ we have the error equation

$$e^2 = A^{-1}(B - w_1I)e^1 + w_1A^{-1}e^0 + A^{-1}R$$

If we take the norm of this equality, we obtain

$$\begin{aligned} \|e^2\| &= \|A^{-1}(B - w_1I)\| \|e^1\| + \|A^{-1}R\| \\ &\leq [\|A^{-1}(B - w_1I)\| + \|A^{-1}w_1\|] w_1^{-1} \|R\| \\ &\leq w_1^{-1} \|R\|. \end{aligned}$$

Assume the inequality $\|e^s\| \leq w_{s-1}^{-1} \|R\|$ is true for all $s \leq k$. Now, we will prove that it is also true for $s = k + 1$. We have the following error equation

$$e^{k+1} = A^{-1} \left(Be^k - w_1e^k + \sum_{m=1}^{k-1} (w_{k-m} - w_{k-m+1}) e^m + w_ke^0 + R \right)$$

$$\begin{aligned} \|e^{k+1}\| &\leq \|A^{-1}(B - w_1I)\| \|e^k\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1}|w_{k-m} - w_{k-m+1}|\| \|e^m\| + \|A^{-1}R\| \\ &\leq \|A^{-1}(B - w_1I)\| w_{k-1}^{-1} \|R\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1}|w_{k-m} - w_{k-m+1}|\| w_{m-1}^{-1} \|R\| + \|A^{-1}w_k\| w_k^{-1} \|R\| \\ &\leq \|A^{-1}(B - w_1I)\| w_k^{-1} \|R\| \\ &\quad + \sum_{m=1}^{k-1} \|A^{-1}|w_{k-m} - w_{k-m+1}|\| w_k^{-1} \|R\| + \|A^{-1}w_k\| w_k^{-1} \|R\| \\ &\leq \left(\|A^{-1}(B - w_1I)\| + \sum_{m=1}^{k-1} \|A^{-1}|w_{k-m} - w_{k-m+1}|\| + \|A^{-1}w_k\| \right) w_k^{-1} \|R\| \\ &\leq (\|A^{-1}(B - w_1I)\| + \|A^{-1}w_1\|) w_k^{-1} \|R\| \\ &\leq w_k^{-1} \|R\| \end{aligned}$$

Since,

$$\lim_{k \rightarrow \infty} \frac{k^{-\alpha}}{(k + \frac{1}{2})^{1-\alpha} - (k - \frac{1}{2})^{1-\alpha}} = \frac{1}{1-\alpha}$$

there exists constant $C > 0$ such that

$$\begin{aligned} \|e^{k+1}\| &\leq w_k^{-1} \|R\| = k^\alpha k^{-\alpha} w_k^{-1} \|R\| = \frac{(k^\alpha \tau^\alpha) k^{-\alpha} \Gamma(2 - \alpha)}{(k + \frac{1}{2})^{1-\alpha} - (k - \frac{1}{2})^{1-\alpha}} \|R\| \\ &\leq C \frac{\Gamma(2 - \alpha)}{1 - \alpha} (\tau^{2-\alpha} + h^2) \\ &\leq Z(\alpha) (\tau^{2-\alpha} + h^2), \end{aligned}$$

where $Z(\alpha) = C \frac{\Gamma(2-\alpha)}{1-\alpha}$.

6 Numerical Example

$$\begin{cases} \frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \frac{\partial^2 u(x,t)}{\partial x^2} + x(1-x) \frac{6t^{3-1/2}}{\Gamma(4-1/2)} + 2(t^3 + 1), \\ (0 < x < 1, 0 < t < 1), \\ u(x, 0) = (1-x)x, \quad 0 \leq x \leq 1, \\ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq 1. \end{cases}$$

Exact solution of this problem is $u(x, t) = (1-x)x(t^3 + 1)$. The solution by the proposed scheme is given in Figure 1. The errors when solving this problem are listed in the Table 1 for various values of time and space nodes.

Figure 1: The errors for some values of M and N when $t = 1$.

Table 1: Error table

Space Nodes(M)	Time Nodes(N)	Error
32	4	0.0124648486
32	8	0.0032352895
32	16	0.0008536728
32	32	0.0002291988
32	64	0.0000628150
32	128	0.0000176561

7 Conclusion

The matrix stability of the difference scheme for time-fractional heat problems is proved. Convergence of the finite difference scheme is showed. Numerical experiments are carried out to support the theoretical claims. These techniques can also be extended to analyze other fractional partial differential equations.

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