Split Equality Fixed Point Problem
for Strictly Pseudocontractive Mappings

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Abstract

In this paper, we introduce a new algorithm for solving the split equality fixed point problem of strictly pseudocontractive mapping in the framework of infinite-dimensional real Hilbert spaces. The strong and weak convergence theorems are obtained. Our results presented in this paper improve and extend some recent corresponding results.

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1 Introduction

For modeling inverse problems which arise from phase retrievals and in medical

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image reconstruction [1], in 1994 Censor and Elfving [2] first introduced the following split feasibility problem (SFP) in finite-dimensional Hilbert spaces:

Let $C$ and $Q$ be the nonempty closed convex subsets of the Hilbert spaces $H_1$ and $H_2$, respectively, $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point $x^*$ with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q.$$

(1.1)

The SFP has been found that it can be used in many areas such as image restoration, computer tomograph, and radiation therapy treatment planing [3-5]. Some methods have been proposed to solve split feasibility problem, see for instance [1,6-9].

Assuming that the SFP is consistent (i.e.(1.1)) has a solution), it is not hard to see that

$$x^* = P_C(I + \gamma A^*(P_Q - I)Ax^*), \forall x \in C,$$

(1.2)

where $P_C$ and $P_Q$ are the (orthogonal) projection onto $C$ and $Q$, respectively, $\gamma > 0$, and $A^*$ denotes the adjoint of $A$. That is, $x^*$ solves the SFP (1.1) if and only if $x^*$ solves the fixed point equation (1.2) [see 10]. This implies that we can use fixed point algorithms to solve SFP.

Recently, Moudafi [11] introduced the following new split feasibility problems:

Let $H_1, H_2, H_3$ be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets, $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. The new split feasibility problem is to find $x^* \in C, y^* \in Q$ such that $Ax^* = By^*$.

(1.3)

This allows asymmetric and partial relations between the variables $x$ and $y$.

It is easy to see that the problem (1.3) reduces to the problem (1.1) as $H_2 = H_3$ and $B = I$ ($I$ stands for the identity mapping from $H_2$ to $H_2$) in (1.3). Therefore the new split feasibility problem (1.3) proposed by Moudafi is generalization of the split feasibility problem (1.1).

Since each nonempty closed convex subset of a Hilbert space can be regards as a set of fixed points of a projection, in [12], Moudafi proposed a new split feasibility problem, which is also called split equality fixed point problem, that is,

$$\text{Find } x \in C, y \in Q \text{ such that } Ax = By, \text{ with } C := \text{Fix}U, Q := \text{Fix}T,$$

(1.4)

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear operators with nonempty fixed point sets $C := \text{Fix}U, Q := \text{Fix}T, A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. This allows asymmetric and partial relations between the variables $x$ and $y$. The interest is to cover many situations, for instance in...
decomposition methods for PDE’s, applications in game theory and in intensity-modulated radiation therapy.

We use $\Omega$ to denote the set of solutions of the new split feasibility problem (1.4), i.e.,

$$\Omega = \{(x, y) : Ax = By, x \in C, y \in Q\}.$$  

To solve the (1.4), Moudafi [12] presented the following simultaneous iterative method and obtained weak convergence theorem:

$$\begin{align*}
&x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)); \\
y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)),
\end{align*}$$  

where $H_1$, $H_2$, $H_3$ are real Hilbert spaces, $U : H_1 \to H_1$, $T : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, $A^*$ and $B^*$ are the adjoint of $A$ and $B$, respectively, $C$ is the set of fixed points of $U$ and $Q$ is the set of fixed points of $T$.

Motivated by the work of Moudafi [11, 12], in this paper, we construct the following iterative algorithm to solve the split equality fixed point problem of strictly pseudocontractive mappings in real Hilbert spaces:

$$\begin{align*}
&\forall x_1 \in H_1, \forall y_1 \in H_2; \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\
y_{n+1} = \alpha_n y_n + (1 - \alpha_n)T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1.
\end{align*}$$  

and to obtain the strong and weak convergence theorems for strictly pseudocontractive mapping which is more general than firmly quasi-nonexpansive mapping and nonexpansive mapping. Our results extend and improve the corresponding results of Moudafi [11,12].

\section{Preliminaries}

We first recall some definitions and lemmas which will be needed in proving our main results.

\textbf{Definition 2.1.} Let $H$ be a Hilbert space.

(1) A single-value mapping $T : H \to H$ is said to be demi-closed at origin, if for any sequence $\{x_n\} \subset H$ with $x_n \to x^*$, and $\|(I - T)x_n\| \to 0$, then we have $x^* = Tx^*$.

(2) A single-value mapping $T : H \to H$ is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset H$ with $\|(I - T)x_n\| \to 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $x^* \in H$. 

**Definition 2.2.** Let \( H \) be a real Hilbert space.

(1) A mapping \( T : H \rightarrow H \) is said to be nonexpansive if
\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.
\] (2.1)

(2) A mapping \( T : H \rightarrow H \) is said to be quasi-nonexpansive if
\[
\|Tx - q\| \leq \|x - q\|, \quad \forall (x, q) \in H \times F(T).
\] (2.2)

(3) A mapping \( T : H \rightarrow H \) is said to be \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that
\[
\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.
\] (2.3)

(4) A mapping \( T : H \rightarrow H \) is said to be \( k \)-strictly pseudocontractive if there exists \( k \in [0, 1) \), such that
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \text{for } x, y \in H. \quad (2.4)
\]

It is obvious that \( T \) is nonexpansive if and only if \( T \) is a 0-strict pseudocontraction. So, the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

**Lemma 2.3.**[13] Let \( C \) be a nonempty closed convex subset of \( H \) and let \( T : C \rightarrow C \) be a \( k \)-strictly pseudocontractive mapping, then the following results hold:

(1) equation (2.4) is equivalent to
\[
<Tx - Ty, x - y> \leq \|x - y\|^2 - \frac{1 - k}{2}\|x - y - (Tx - Ty)\|^2, \quad \forall x, y \in C. \quad (2.5)
\]

(2) \( T \) is Lipschitz continuous with a constant \( \frac{1 + k}{1 - k} \), i.e.,
\[
\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|, \quad \forall x, y \in C. \quad (2.6)
\]

(3) (Demi-closed principle) \( I - T \) is demi-closed on \( C \), that is,
\[
if \ x_n \to x^* \in C \ and \ (I - T)x_n \to 0, \ then \ x^* = Tx^*. \quad (2.7)
\]
It is clear that equation (2.4) is equivalent to
\[
(I - T)x - (I - T)y, x - y \geq \frac{1 - k}{2} \| (I - T)x - (I - T)y \|^2, \quad \forall x, y \in C. \quad (2.8)
\]

**Lemma 2.4.** [12] Let $H$ be a Hilbert space and $\{\mu_n\}$ be a sequence in $H$ such that there exists a nonempty set $W \subset H$ satisfying:

(i) For every $\mu^* \in W$, $\lim_{n \to \infty} \| \mu_n - \mu^* \|$ exists.

(ii) Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to $W$.

Then, there exists $\mu^* \in W$ such that $\{\mu_n\}$ weakly converges to $\mu^*$.

**Lemma 2.5** ([14]) Let $H$ be a real Hilbert space, then for all $x, y \in H$, we have
\[
\| x - y \|^2 \leq \| x \|^2 - \| y \|^2 - 2 < x - y, y >. \quad (2.9)
\]

### 3 Main result

**Theorem 3.1.** Let $H_1, H_2, H_3$ be real Hilbert spaces, let $T_1 : H_1 \to H_1$, $T_2 : H_2 \to H_2$ be two strictly pseudocontractive mappings for some $k \in [0, 1)$, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

\[
\begin{align*}
\forall x_1 \in H_1, & \quad \forall y_1 \in H_2; \\
x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T_1 (x_n - \gamma_n A^*(Ax_n - By_n)); \\
y_{n+1} &= \alpha_n y_n + (1 - \alpha_n) T_2 (y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1.
\end{align*}
\]

where $\lambda_A$ and $\lambda_B$ stand for the spectral radius of $A^*A$ and $B^*B$ respectively, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\frac{k + 1}{2} < \alpha \leq \alpha_n \leq \beta < 1$ (for some $\alpha, \beta \in (0, 1)$), and $\{\gamma_n\}$ is a positive real sequence such that $\gamma_n \in (\varepsilon, \frac{2(1 - \beta)}{(1 - \alpha)^2 + \beta^2(\lambda_A + \lambda_B)} - \varepsilon)$ (for $\varepsilon$ small enough), $C := F(T_1)$ and $Q := F(T_2)$. If $\Omega \neq \emptyset$, then

(I) The sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x, y)$ of the problem (1.4).

(II) In addition, if $T_1, T_2$ are also semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of the problem (1.4).

**Proof:**

Now we prove the conclusion (I).
Let \((x, y) \in \Omega\). Since \(\|\cdot\|^2\) is convex and \(T_1, T_2\) are strictly pseudocontractive mappings, we have

\[
\|x_{n+1} - x\|^2 = \|\alpha_n x_n + (1 - \alpha_n)T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2
\]

\[
= \alpha_n^2\|x_n - x\|^2 + (1 - \alpha_n)^2\|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2
\]

\[
+ 2\alpha_n(1 - \alpha_n)\langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle - x > \leq\alpha_n^2\|x_n - x\|^2 + (1 - \alpha_n)^2\|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2
\]

\[
+ (1 - \alpha_n)^2k\|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2
\]

\[
+ 2\alpha_n(1 - \alpha_n)\langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle - x > .
\]

By (2.8), we have

\[
\langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle - x > =\langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle - (x_n - \gamma_n A^*(Ax_n - By_n))
\]

\[
+ x_n - \gamma_n A^*(Ax_n - By_n) - x >
\]

\[
=\langle x_n - x, (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle >\langle x_n - x_n - x, x_n - \gamma_n A^*(Ax_n - By_n) \rangle
\]

\[
=\langle x_n - \gamma_n A^*(Ax_n - By_n) - x + \gamma_n A^*(Ax_n - By_n) , (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle
\]

\[
+ \|x_n - x\|^2 - \gamma_n < Ax_n - Ax, Ax_n - By_n >
\]

\[
\leq -\frac{1 - k}{2}\|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2
\]

\[
+ \|x_n - x\|^2 - \gamma_n < Ax_n - Ax, Ax_n - By_n >
\]

\[
\leq -\frac{1 - k}{2}\|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2
\]

\[
+ \|x_n - x\|^2 - \gamma_n < Ax_n - Ax, Ax_n - By_n > .
\]

On the other hand, since

\[
\|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2
\]

\[
= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 - 2\gamma_n < x_n - x, A^*(Ax_n - By_n) >
\]

\[
= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 - 2\gamma_n < Ax_n - Ax, Ax_n - By_n > ,
\]

(3.3)
and
\[
\|\gamma_n A^*(Ax_n - By_n)\|^2 = \gamma_n^2 < A^*(Ax_n - By_n), A^*(Ax_n - By_n) > \\
= \gamma_n^2 < Ax_n - By_n, AA^*(Ax_n - By_n) > \\
\leq \lambda A\gamma_n^2 < Ax_n - By_n, Ax_n - By_n > \\
= \lambda A\gamma_n^2 \|Ax_n - By_n\|^2. 
\]

Combine (3.1), (3.2), (3.3) and (3.4), then we have
\[
\|x_{n+1} - x\|^2 \\
\leq \alpha_n^2 \|x_n - x\|^2 + (1 - \alpha_n)^2 \{\|x_n - x\|^2 - 2\gamma_n < Ax_n - Ax, Ax_n - By_n > \\
+ \|\gamma_n A^*(Ax_n - By_n)\|^2 + k\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2\} \\
+ 2\alpha_n(1 - \alpha_n)\{\{1 - k\} \alpha_n(1 - \alpha_n) + k(1 - \alpha_n)^2\}\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
+ \|\gamma_n A^*(Ax_n - By_n)\|\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\| \\
+ \|x_n - x\|^2 - \gamma_n < Ax_n - Ax, Ax_n - By_n > \}
= \{\alpha_n^2 + 2\alpha_n(1 - \alpha_n) + (1 - \alpha_n)^2\}\|x_n - x\|^2 \\
+ \{1 - \{1 - k\} \alpha_n(1 - \alpha_n) + k(1 - \alpha_n)^2\}\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
+ 2\alpha_n(1 - \alpha_n)\|\gamma_n A^*(Ax_n - By_n)\|\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\| \\
- \{2\alpha_n(1 - \alpha_n) + 2(1 - \alpha_n)^2\}\gamma_n < Ax_n - Ax, Ax_n - By_n > \\
+ (1 - \alpha_n)^2 \|\gamma_n A^*(Ax_n - By_n)\|^2
\]
\[
\leq \|x_n - x\|^2 + (1 - \alpha_n)(k - \alpha_n)\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
+ \{\alpha_n^2\|\gamma_n A^*(Ax_n - By_n)\|^2 + (1 - \alpha_n)^2\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2\} \\
- \{2\alpha_n(1 - \alpha_n) + 2(1 - \alpha_n)^2\}\gamma_n < Ax_n - Ax, Ax_n - By_n > \\
+ (1 - \alpha_n)^2 \|\gamma_n A^*(Ax_n - By_n)\|^2 \\
= \|x_n - x\|^2 + \{1 - \alpha_n\}(k + 1 - 2\alpha_n)\|\|I - T_1\|(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
- 2(1 - \alpha_n)\gamma_n < Ax_n - Ax, Ax_n - By_n > \\
+ ((1 - \alpha_n)^2 + \alpha_n^2)\|\gamma_n A^*(Ax_n - By_n)\|^2. 
\]

Similarly, from the second equality of algorithm we can get
\[
\|y_{n+1} - y\|^2 = \|y_n - x\|^2 + \{1 - \alpha_n\}(k + 1 - 2\alpha_n)\|\|I - T_2\|(y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
+ 2(1 - \alpha_n)\gamma_n < By_n - By, Ax_n - By_n > \\
+ ((1 - \alpha_n)^2 + \alpha_n^2)\|\gamma_n B^*(Ax_n - By_n)\|^2. 
\]
Since \((x, y) \in \Omega\) so we have the fact that \(Ax = By\), and finally we have

\[
\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \\
\leq \|x_n - x\|^2 + \|y_n - y\|^2 \\
- \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
- \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
- \gamma_n\{2(1 - \alpha_n) - ((1 - \alpha_n)^2 + \alpha_n^2)\gamma_n(\lambda_A + \lambda_B)\}\|Ax_n - By_n\|^2
\]

Let \(\Omega_n(x, y) := \|x_n - x\|^2 + \|y_n - y\|^2\) then we have

\[
\Omega_{n+1}(x, y) \leq \Omega_n(x, y) - \gamma_n\{2(1 - \beta) - ((1 - \alpha_n)^2 + \beta^2)\gamma_n(\lambda_A + \lambda_B)\}\|Ax_n - By_n\|^2 \\
- \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
- \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2.
\] (3.7)

Obviously the sequence \(\{\Omega_n(x, y)\}\) is decreasing and has lower bounded, so it converges to some finite limit \(\omega(x, y)\). This means that the first condition of Lemma 2.4 (Opial’s lemma) is satisfied with \(W = \Omega\), \(\mu_n := (x_n, y_n)\), \(\mu^* := (x, y)\). And by passing to limit in (3.8), we obtain that

\[
\lim_{n \to \infty} \|Ax_n - By_n\| = 0,
\] (3.9)

and

\[
\lim_{n \to \infty} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\| = 0, \quad (3.10)
\]

\[
\lim_{n \to \infty} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\| = 0. \quad (3.11)
\]

Since \(\|x_n - x\| \leq \Omega_n(x, y)\), \(\|y_n - y\| \leq \Omega_n(x, y)\) and \(\lim_{n \to \infty} \Omega_n(x, y)\) exists, we know that \(\{x_n\}\) and \(\{y_n\}\) are bounded, and \(\limsup_{n \to \infty} \|x_n - x\|\) and \(\limsup_{n \to \infty} \|y_n - y\|\) exist. Let \(x^*\) and \(y^*\) be respectively weak cluster points of the sequences \(\{x_n\}\) and \(\{y_n\}\), Further, \(\{x_n - \gamma_n A^*(Ax_n - By_n)\}\) also converges weakly to \(x^*\), \(\{y_n + \gamma_n B^*(Ax_n - By_n)\}\) converges weakly to \(y^*\). From Lemma 2.5, we have

\[
\|x_{n+1} - x\|^2 = \|x_{n+1} - x - x_n + x\|^2 \\
= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 < x_{n+1} - x_n, x_n - x > \\
= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 < x_{n+1} - x^*, x_n - x > + 2 < x_n - x^*, x_n - x >.
\] (3.12)
so
\[
\limsup_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\] (3.13)

Similarly
\[
\limsup_{n \to \infty} \|y_{n+1} - y_n\| = 0.
\] (3.14)

Since \(k\)-strictly pseudocontractive mapping Lipschitz continuous with a constant \(1 + \frac{k}{1 - k}\), from Lemma 2.3, we obtain
\[
\|x_n - T_1 x_n\| = \|x_n - x_{n+1} + x_{n+1} - T_1 x_n\|
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1 x_n\|
= \|x_n - x_{n+1}\| + \|\alpha_n x_n + (1 - \alpha_n) T_1(x_n - \gamma_n A^* (Ax_n - By_n)) - T_1 x_n\|
\leq \|x_n - x_{n+1}\| + \alpha_n \|x_n - T_1 x_n\|
+ (1 - \alpha_n) \|T_1(x_n - \gamma_n A^* (Ax_n - By_n)) - T_1 x_n\|
\leq \|x_n - x_{n+1}\| + \alpha_n \|x_n - T_1 x_n\| + (1 - \alpha_n) \frac{1 + k}{1 - k} \| - \gamma_n A^* (Ax_n - By_n)\|.
\] (3.15)

It follows from (3.9) and (3.13) that
\[
\lim_{n \to \infty} \|x_n - T_1 x_n\| = 0.
\] (3.16)

Similarly
\[
\lim_{n \to \infty} \|y_n - T_2 y_n\| = 0.
\] (3.17)

Since \(\{x_n\}\) and \(\{y_n\}\) converges weakly to \(x^*\) and \(y^*\), respectively, then it follows from (3.10), (3.11), (3.16), (3.17) and Lemma 2.3 that \(x^* \in F(T_1)\) and \(y^* \in F(T_2)\).

On the other hand, since the squared norm is weakly lower semicontinuous, we have
\[
\|Ax^* - By^*\|^2 \leq \liminf_{n \to \infty} \|Ax_n - B_n\|^2 = 0,
\]
therefore \(Ax^* = By^*\). This implies that \((x^*, y^*) \in \Omega\). Thus from Lemma 2.4, we know that \((x_n, y_n)\) converges weakly to \((x^*, y^*)\). The proof of conclusion(I) is completed.

Next, we prove the conclusion(II).

Since \(T_1\) and \(T_2\) are semi-compact, \(\{x_n\}\) and \(\{y_n\}\) are bounded and \(\lim_{n \to \infty} \|x_n - T_1 x_n\| = 0, \lim_{n \to \infty} \|y_n - T_2 y_n\| = 0\), then there exist subsequences \(\{x_{n_j}\}\) and \(\{y_{n_j}\}\) of \(\{x_n\}\) and \(\{y_n\}\) such that \(\{x_{n_j}\}\) and \(\{y_{n_j}\}\) converge strongly to \(x^*\) and \(y^*\), respectively.

From Lemma 2.3, we have \(x^* \in F(T_1)\) and \(y^* \in F(T_2)\). Further, since the squared norm is weakly lower semicontinuous and \(Ax_{n_j} - By_{n_j} \to Ax^* - By^*\), we have
\[
\|Ax^* - By^*\|^2 \leq \liminf_{j \to \infty} \|Ax_{n_j} - B_{n_j}\|^2 = 0,
\]
so \( Ax^* = By^* \). This implies that \((x^*, y^*) \in \Omega\).

On the other hand, since \( \Omega_n(x, y) = \|x - x_n\| + \|y_n - y\| \) for any \((x, y) \in \Omega\), we know that \( \lim j \to \infty \Omega_n(x^*, y^*) = 0 \). From conclusion(I), we have \( \lim n \to \infty \Omega_n(x^*, y^*) = 0 \). Further, we can obtain that \( \lim n \to \infty \|x - x^*\| = 0 \) and \( \lim n \to \infty \|y_n - y^*\| = 0 \). This completes the proof of the conclusion(II).

For nonexpansive mapping, we have \( k = 0 \) in Theorem 3.1 to obtain the following corollary.

**Corollary 3.2.** Let \( H_1, H_2, H_3 \) be real Hilbert spaces, let \( T_1 : H_1 \to H_1 \), \( T_2 : H_2 \to H_2 \) be two nonexpansive mappings, and \( A : H_1 \to H_3 \), \( B : H_2 \to H_3 \) be two bounded linear operators. Assume that the iteration scheme \( \{(x_n, y_n)\} \) is defined as follows:

\[
\begin{align*}
\forall x_1 \in H_1, \quad &\forall y_1 \in H_2; \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\
y_{n+1} = \alpha_n y_n + (1 - \alpha_n) T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1.
\end{align*}
\]

where \( \lambda_A \) and \( \lambda_B \) stand for the spectral radius of \( A^*A \) and \( B^*B \) respectively, \( \{\alpha_n\} \) is a sequence in \((0, 1)\) with \( \frac{1}{2} < \alpha \leq \alpha_n \leq \beta < 1 \) (for some \( \alpha, \beta \in (0, 1) \)), and \( \{\gamma_n\} \) is a positive real sequence such that \( \gamma_n \in \left( \varepsilon, \frac{2(1 - \beta)}{(1 - \alpha)^2 + \beta^2 (\lambda_A + \lambda_B)} - \varepsilon \right) \) (for \( \varepsilon \) small enough), \( C := F(T_1) \) and \( Q := F(T_2) \). If \( \Omega \neq \emptyset \), then

(I) The sequence \( \{(x_n, y_n)\} \) converges weakly to a solution \((x, y)\) of the problem \((1.4)\).

(II) In addition, if \( T_1, T_2 \) are also semi-compact, then \( \{(x_n, y_n)\} \) converges strongly to a solution of the problem \((1.4)\).

**Remark 3.3.** The iterative method of Theorem 3.1 reduces to the iterative method in \([12]\) when \( \alpha_n = 0 \) for all \( n \geq 1 \). The results obtained in this paper extend the results from firmly quasi-nonexpansive mappings and nonexpansive mappings to more general strictly pseudocontractive mappings. So, our results improve and extend the results of Moudafi \([11,12]\).

**References**


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