Split Equality Fixed Point Problem for Strictly Pseudocontractive Mappings¹

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Abstract

In this paper, we introduce a new algorithm for solving the split equality fixed point problem of strictly pseudocontractive mapping in the framework of infinite-dimensional real Hilbert spaces. The strong and weak convergence theorems are obtained. Our results presented in this paper improve and extend some recent corresponding results.

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1 Introduction

For modeling inverse problems which arise from phase retrievals and in medical

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image reconstruction [1], in 1994 Censor and Elfving [2] first introduced the following split feasibility problem (SFP) in finite-dimensional Hilbert spaces:

Let C and Q be the nonempty closed convex subsets of the Hilbert spaces H_1 and H_2 , respectively, $A : H_1 \rightarrow H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \quad (1.1)$$

The SFP has been found that it can be used in many areas such as image restoration, computer tomograph, and radiation therapy treatment planing [3-5]. Some methods have been proposed to solve split feasibility problem, see for instance [1,6-9].

Assuming that the *SFP* is consistent (i.e.(1.1)) has a solution), it is not hard to see that

$$x^* = P_C(I + \gamma A^*(P_Q - I)Ax^*, \forall x \in C, \quad (1.2)$$

where P_C and P_Q are the (orthogonal) projection onto C and Q , respectively, $\gamma > 0$, and A^* denotes the adjoint of A . That is, x^* solves the *SFP* (1.1) if and only if x^* solves the fixed point equation (1.2) [see 10]. This implies that we can use fixed point algorithms to solve *SFP*.

Recently, Moudafi [11] introduced the following new split feasibility problems:

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets, $A : H_1 \rightarrow H_3, B : H_2 \rightarrow H_3$ be two bounded linear operators. The new split feasibility problem is to

$$\text{find } x^* \in C, y^* \in Q \quad \text{such that} \quad Ax^* = By^*. \quad (1.3)$$

This allows asymmetric and partial relations between the variables x and y .

It is easy to see that the problem (1.3) reduces to the problem (1.1) as $H_2 = H_3$ and $B = I$ (I stands for the identity mapping from H_2 to H_2) in (1.3). Therefore the new split feasibility problem (1.3) proposed by Moudafi is generalization of the split feasibility problem (1.1).

Since each nonempty closed convex subset of a Hilbert space can be regards as a set of fixed points of a projection, in [12], Moudafi proposed a new split feasibility problem, which is also called split equality fixed point problem, that is,

$$\text{Find } x \in C, y \in Q \text{ such that } Ax = By, \text{ with } C := \text{Fix}U, Q := \text{Fix}T, \quad (1.4)$$

where $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ are two nonlinear operators with nonempty fixed point sets $C := \text{Fix}U, Q := \text{Fix}T, A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are two bounded linear operators. This allows asymmetric and partial relations between the variables x and y . The interest is to cover many situations, for instance in

decomposition methods for *PDE's*, applications in game theory and in intensity-modulated radiation therapy.

We use Ω to denote the set of solutions of the new split feasibility problem (1.4), i.e.,

$$\Omega = \{(x, y) : Ax = By, x \in C, y \in Q\}. \quad (1.5)$$

To solve the (1.4), Modaufi [12] presented the following simultaneous iterative method and obtained weak convergence theorem:

$$(SIM - FPP) \begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \quad (1.6)$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \rightarrow H_1, T : H_2 \rightarrow H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \rightarrow H_3, B : H_2 \rightarrow H_3$ are two bounded linear operators, A^* and B^* are the adjoint of A and B , respectively, C is the set of fixed points of U and Q is the set of fixed points of T .

Motivated by the work of Moudafi [11, 12], in this paper, we construct the following iterative algorithm to solve the split equality fixed point problem of strictly pseudocontractive mappings in real Hilbert spaces:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n)T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1. \end{cases} \quad (1.7)$$

and to obtain the strong and weak convergence theorems for strictly pseudocontractive mapping which is more general than firmly quasi-nonexpansive mapping and nonexpansive mapping. Our results extend and improve the corresponding results of Moudafi [11,12].

2 Preliminaries

We first recall some definitions and lemmas which will be needed in proving our main results.

Definition 2.1. Let H be a Hilbert space.

- (1) A single-value mapping $T : H \rightarrow H$ is said to be demi-closed at origin, if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $\|(I - T)x_n\| \rightarrow 0$, then we have $x^* = Tx^*$.
- (2) A single-value mapping $T : H \rightarrow H$ is said to be semi-compact, if for any bounded sequence $\{x_n\} \subset H$ with $\|(I - T)x_n\| \rightarrow 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $x^* \in H$.

Definition 2.2. Let H be a real Hilbert space.

- (1) A mapping $T : H \rightarrow H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \quad (2.1)$$

- (2) A mapping $T : H \rightarrow H$ is said to be quasi-nonexpansive if

$$\|Tx - q\| \leq \|x - q\|, \quad \forall (x, q) \in H \times F(T). \quad (2.2)$$

- (3) A mapping $T : H \rightarrow H$ is said to be L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H. \quad (2.3)$$

- (4) A mapping $T : H \rightarrow H$ is said to be k -strictly pseudocontractive if there exists $k \in [0, 1)$, such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \text{for } x, y \in H. \quad (2.4)$$

It is obvious that T is nonexpansive if and only if T is a 0-strict pseudocontraction. So, the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

Lemma 2.3.[13] Let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be a k -strictly pseudocontractive mapping, then the following results hold:

- (1) equation (2.4) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2} \|x - y - (Tx - Ty)\|^2, \quad \forall x, y \in C. \quad (2.5)$$

- (2) T is Lipschitz continuous with a constant $\frac{1+k}{1-k}$, i.e.,

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k} \|x - y\|, \quad \forall x, y \in C. \quad (2.6)$$

- (3) (Demi-closed principle) $I - T$ is demi-closed on C , that is,

$$\text{if } x_n \rightharpoonup x^* \in C \text{ and } (I - T)x_n \rightarrow 0, \text{ then } x^* = Tx^*. \quad (2.7)$$

It is clear that equation (2.4) is equivalent to

$$\langle (I - T)x - (I - T)y, x - y \rangle \geq \frac{1 - k}{2} \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \quad (2.8)$$

Lemma 2.4.[12] Let H be a Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying:

- (i) For every $\mu^* \in W$, $\lim_{n \rightarrow \infty} \|\mu_n - \mu^*\|$ exists.
- (ii) Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W .

Then, there exists $\mu^* \in W$ such that $\{\mu_n\}$ weakly converges to μ^* .

Lemma 2.5 ([14]) Let H be a real Hilbert space, then for all $x, y \in H$, we have

$$\|x - y\|^2 \leq \|x\|^2 - \|y\|^2 - 2 \langle x - y, y \rangle. \quad (2.9)$$

3 Main result

Theorem 3.1. Let H_1, H_2, H_3 be real Hilbert spaces, let $T_1 : H_1 \rightarrow H_1$, $T_2 : H_2 \rightarrow H_2$ be two strictly pseudocontractive mappings for some $k \in [0, 1)$, and $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ be two bounded linear operators. Assume that the iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n) T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1. \end{cases}$$

where λ_A and λ_B stand for the spectral radius of A^*A and B^*B respectively, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\frac{k+1}{2} < \alpha \leq \alpha_n \leq \beta < 1$ (for some $\alpha, \beta \in (0, 1)$), and $\{\gamma_n\}$ is a positive real sequence such that $\gamma_n \in (\varepsilon, \frac{2(1-\beta)}{((1-\alpha)^2 + \beta^2)(\lambda_A + \lambda_B)} - \varepsilon)$ (for ε small enough), $C := F(T_1)$ and $Q := F(T_2)$. If $\Omega \neq \emptyset$, then

- (I) The sequence $\{(x_n, y_n)\}$ converges weakly to a solution (x, y) of the problem (1.4).
- (II) In addition, if T_1, T_2 are also semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of the problem (1.4).

Proof:

Now we prove the conclusion (I).

Let $(x, y) \in \Omega$. Since $\|\cdot\|^2$ is convex and T_1, T_2 are strictly pseudocontractive mappings, we have

$$\begin{aligned}
 \|x_{n+1} - x\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\
 &= \alpha_n^2 \|x_n - x\|^2 + (1 - \alpha_n)^2 \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\
 &\quad + 2\alpha_n(1 - \alpha_n) \langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x \rangle \\
 &\leq \alpha_n^2 \|x_n - x\|^2 + (1 - \alpha_n)^2 \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \\
 &\quad + (1 - \alpha_n)^2 k \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
 &\quad + 2\alpha_n(1 - \alpha_n) \langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x \rangle.
 \end{aligned} \tag{3.1}$$

By (2.8), we have

$$\begin{aligned}
 &\langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - x \rangle \\
 &= \langle x_n - x, T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - (x_n - \gamma_n A^*(Ax_n - By_n)) \\
 &\quad + x_n - \gamma_n A^*(Ax_n - By_n) - x \rangle \\
 &= \langle x_n - x, (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle \\
 &\quad + \langle x_n - x, x_n - x - \gamma_n A^*(Ax_n - By_n) \rangle \\
 &= \langle x_n - \gamma_n A^*(Ax_n - By_n) - x + \gamma_n A^*(Ax_n - By_n) \\
 &\quad, (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle \\
 &\quad + \|x_n - x\|^2 - \langle x_n - x, \gamma_n A^*(Ax_n - By_n) \rangle \\
 &= \langle x_n - \gamma_n A^*(Ax_n - By_n) - x, (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle \\
 &\quad + \langle \gamma_n A^*(Ax_n - By_n), (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle \\
 &\quad + \|x_n - x\|^2 - \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
 &\leq -\frac{1-k}{2} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
 &\quad + \langle \gamma_n A^*(Ax_n - By_n), (T_1 - I)(x_n - \gamma_n A^*(Ax_n - By_n)) \rangle \\
 &\quad + \|x_n - x\|^2 - \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
 &\leq -\frac{1-k}{2} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
 &\quad + \|\gamma_n A^*(Ax_n - By_n)\| \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\| \\
 &\quad + \|x_n - x\|^2 - \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle.
 \end{aligned} \tag{3.2}$$

On the other hand, since

$$\begin{aligned}
 &\|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \\
 &= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 - 2\gamma_n \langle x_n - x, A^*(Ax_n - By_n) \rangle \\
 &= \|x_n - x\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 - 2\gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle,
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\|\gamma_n A^*(Ax_n - By_n)\|^2 &= \gamma_n^2 \langle A^*(Ax_n - By_n), A^*(Ax_n - By_n) \rangle \\
&= \gamma_n^2 \langle Ax_n - By_n, AA^*(Ax_n - By_n) \rangle \\
&\leq \lambda_A \gamma_n^2 \langle Ax_n - By_n, Ax_n - By_n \rangle \\
&= \lambda_A \gamma_n^2 \|Ax_n - By_n\|^2.
\end{aligned} \tag{3.4}$$

Combine (3.1), (3.2), (3.3) and (3.4), then we have

$$\begin{aligned}
&\|x_{n+1} - x\|^2 \\
&\leq \alpha_n^2 \|x_n - x\|^2 + (1 - \alpha_n)^2 \{ \|x_n - x\|^2 - 2\gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&\quad + \|\gamma_n A^*(Ax_n - By_n)\|^2 + k \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \} \\
&\quad + 2\alpha_n(1 - \alpha_n) \left\{ -\frac{1-k}{2} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \right. \\
&\quad + \|\gamma_n A^*(Ax_n - By_n)\| \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\| \\
&\quad + \|x_n - x\|^2 - \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \} \\
&= \{\alpha_n^2 + 2\alpha_n(1 - \alpha_n) + (1 - \alpha_n)^2\} \|x_n - x\|^2 \\
&\quad + \left\{ -2\left(\frac{1-k}{2}\right) \alpha_n(1 - \alpha_n) + k(1 - \alpha_n)^2 \right\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n) \|\gamma_n A^*(Ax_n - By_n)\| \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\| \\
&\quad - \{2\alpha_n(1 - \alpha_n) + 2(1 - \alpha_n)^2\} \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&\quad + (1 - \alpha_n)^2 \|\gamma_n A^*(Ax_n - By_n)\|^2 \\
&\leq \|x_n - x\|^2 + (1 - \alpha_n)(k - \alpha_n) \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad + \{\alpha_n^2 \|\gamma_n A^*(Ax_n - By_n)\|^2 + (1 - \alpha_n)^2 \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2\} \\
&\quad - \{2\alpha_n(1 - \alpha_n) + 2(1 - \alpha_n)^2\} \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&\quad + (1 - \alpha_n)^2 \|\gamma_n A^*(Ax_n - By_n)\|^2 \\
&= \|x_n - x\|^2 + \{(1 - \alpha_n)(k + 1 - 2\alpha_n)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
&\quad - 2(1 - \alpha_n) \gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle \\
&\quad + ((1 - \alpha_n)^2 + \alpha_n^2) \|\gamma_n A^*(Ax_n - By_n)\|^2.
\end{aligned} \tag{3.5}$$

Similarly, from the second equality of algorithm we can get

$$\begin{aligned}
\|y_{n+1} - y\|^2 &= \|y_n - x\|^2 + \{(1 - \alpha_n)(k + 1 - 2\alpha_n)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
&\quad + 2(1 - \alpha_n) \gamma_n \langle By_n - By, Ax_n - By_n \rangle \\
&\quad + ((1 - \alpha_n)^2 + \alpha_n^2) \|\gamma_n B^*(Ax_n - By_n)\|^2.
\end{aligned} \tag{3.6}$$

Since $(x, y) \in \Omega$ so we have the fact that $Ax = By$, and finally we have

$$\begin{aligned}
& \|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \\
& \leq \|x_n - x\|^2 + \|y_n - y\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
& \quad - \gamma_n \{2(1 - \alpha_n) - ((1 - \alpha_n)^2 + \alpha_n^2) \gamma_n (\lambda_A + \lambda_B)\} \|Ax_n - By_n\|^2 \\
& \leq \|x_n - x\|^2 + \|y_n - y\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2 \\
& \quad - \gamma_n \{2(1 - \beta) - ((1 - \alpha)^2 + \beta^2) \gamma_n (\lambda_A + \lambda_B)\} \|Ax_n - By_n\|^2.
\end{aligned} \tag{3.7}$$

Let $\Omega_n(x, y) := \|x_n - x\|^2 + \|y_n - y\|^2$ then we have

$$\begin{aligned}
\Omega_{n+1}(x, y) & \leq \Omega_n(x, y) - \gamma_n \{2(1 - \beta) - ((1 - \alpha)^2 + \beta^2) \gamma_n (\lambda_A + \lambda_B)\} \|Ax_n - By_n\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\|^2 \\
& \quad - \{(1 - \alpha_n)(2\alpha_n - k - 1)\} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\|^2.
\end{aligned} \tag{3.8}$$

Obviously the sequence $\{\Omega_n(x, y)\}$ is decreasing and has lower bounded, so it converges to some finite limit $\omega(x, y)$. This means that the first condition of Lemma 2.4 (Opial's lemma) is satisfied with $W = \Omega$, $\mu_n := (x_n, y_n)$, $\mu^* := (x, y)$. And by passing to limit in (3.8), we obtain that

$$\lim_{n \rightarrow \infty} \|Ax_n - By_n\| = 0, \tag{3.9}$$

and

$$\lim_{n \rightarrow \infty} \|(I - T_1)(x_n - \gamma_n A^*(Ax_n - By_n))\| = 0, \tag{3.10}$$

$$\lim_{n \rightarrow \infty} \|(I - T_2)(y_n + \gamma_n B^*(Ax_n - By_n))\| = 0. \tag{3.11}$$

Since $\|x_n - x\| \leq \Omega_n(x, y)$, $\|y_n - y\| \leq \Omega_n(x, y)$ and $\lim_{n \rightarrow \infty} \Omega_n(x, y)$ exists, we know that $\{x_n\}$ and $\{y_n\}$ are bounded, and $\limsup_{n \rightarrow \infty} \|x_n - x\|$ and $\limsup_{n \rightarrow \infty} \|y_n - y\|$ exist. Let x^* and y^* be respectively weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, Further, $\{x_n - \gamma_n A^*(Ax_n - By_n)\}$ also converges weakly to x^* , $\{y_n + \gamma_n B^*(Ax_n - By_n)\}$ converges weakly to y^* . From Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 & = \|x_{n+1} - x - x_n + x\|^2 \\
& = \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_{n+1} - x, x_n - x \rangle \\
& = \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_{n+1} - x^*, x_n - x \rangle + 2 \langle x_n - x^*, x_n - x \rangle,
\end{aligned} \tag{3.12}$$

so

$$\limsup_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.13)$$

Similarly

$$\limsup_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.14)$$

Since k -strictly pseudocontractive mapping Lipschitz continuous with a constant $\frac{1+k}{1-k}$, from Lemma 2.3, we obtain

$$\begin{aligned} \|x_n - T_1 x_n\| &= \|x_n - x_{n+1} + x_{n+1} - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1 x_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n x_n + (1 - \alpha_n)T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|x_n - T_1 x_n\| \\ &\quad + (1 - \alpha_n) \|T_1(x_n - \gamma_n A^*(Ax_n - By_n)) - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n \|x_n - T_1 x_n\| + (1 - \alpha_n) \frac{1+k}{1-k} \|\gamma_n A^*(Ax_n - By_n)\|. \end{aligned} \quad (3.15)$$

It follows from (3.9) and (3.13) that

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0. \quad (3.16)$$

Similarly

$$\lim_{n \rightarrow \infty} \|y_n - T_2 y_n\| = 0. \quad (3.17)$$

Since $\{x_n\}$ and $\{y_n\}$ converges weakly to x^* and y^* , respectively, then it follows from (3.10), (3.11), (3.16), (3.17) and Lemma 2.3 that $x^* \in F(T_1)$ and $y^* \in F(T_2)$.

On the other hand, since the squared norm is weakly lower semicontinuous, we have

$$\|Ax^* - By^*\|^2 \leq \liminf_{n \rightarrow \infty} \|Ax_n - By_n\|^2 = 0,$$

therefore $Ax^* = By^*$. This implies that $(x^*, y^*) \in \Omega$. Thus from Lemma 2.4, we know that $\{(x_n, y_n)\}$ converges weakly to (x^*, y^*) . The proof of conclusion(I) is completed.

Next, we prove the conclusion(II).

Since T_1 and T_2 are semi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$, $\lim_{n \rightarrow \infty} \|y_n - T_2 y_n\| = 0$, then there exist subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ of $\{x_n\}$ and $\{y_n\}$ such that $\{x_{n_j}\}$ and $\{y_{n_j}\}$ converge strongly to x^* and y^* , respectively. From Lemma 2.3, we have $x^* \in F(T_1)$ and $y^* \in F(T_2)$. Further, since the squared norm is weakly lower semicontinuous and $Ax_{n_j} - By_{n_j} \rightarrow Ax^* - By^*$, we have

$$\|Ax^* - By^*\|^2 \leq \liminf_{j \rightarrow \infty} \|Ax_{n_j} - By_{n_j}\|^2 = 0,$$

so $Ax^* = By^*$. This implies that $(x^*, y^*) \in \Omega$.

On the other hand, since $\Omega_n(x, y) = \|x_n - x\|^2 + \|y_n - y\|^2$ for any $(x, y) \in \Omega$, we know that $\lim_{j \rightarrow \infty} \Omega_{n_j}(x^*, y^*) = 0$. From conclusion(I), we have $\lim_{n \rightarrow \infty} \Omega_n(x^*, y^*)$ exists, therefore $\lim_{n \rightarrow \infty} \Omega_n(x^*, y^*) = 0$. Further, we can obtain that $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$. This completes the proof of the conclusion(II).

For nonexpansive mapping, we have $k = 0$ in Theorem 3.1 to obtain the following corollary.

Corollary 3.2. Let H_1, H_2, H_3 be real Hilbert spaces, let $T_1 : H_1 \rightarrow H_1$, $T_2 : H_2 \rightarrow H_2$ be two nonexpansive mappings, and $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ be two bounded linear operators. Assume that the iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_1(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n) T_2(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1. \end{cases}$$

where λ_A and λ_B stand for the spectral radius of A^*A and B^*B respectively, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\frac{1}{2} < \alpha \leq \alpha_n \leq \beta < 1$ (for some $\alpha, \beta \in (0, 1)$), and $\{\gamma_n\}$ is a positive real sequence such that $\gamma_n \in (\varepsilon, \frac{2(1-\beta)}{((1-\alpha)^2 + \beta^2)(\lambda_A + \lambda_B)} - \varepsilon)$ (for ε small enough), $C := F(T_1)$ and $Q := F(T_2)$. If $\Omega \neq \emptyset$, then

- (I) The sequence $\{(x_n, y_n)\}$ converges weakly to a solution (x, y) of the problem (1.4).
- (II) In addition, if T_1, T_2 are also semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of the problem (1.4).

Remark 3.3. The iterative method of Theorem 3.1 reduces to the iterative method in [12] when $\alpha_n = 0$ for all $n \geq 1$. The results obtained in this paper extend the results from firmly quasi-nonexpansive mappings and nonexpansive mappings to more general strictly pseudocontractive mappings. So, our results improve and extend the results of Moudafi [11,12].

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