Disconvergent and Divergent Fuzzy Sequences

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Abstract

We introduce the new concept disconvergent fuzzy sequences in a metric space. Also we define divergent fuzzy sequence in R.

Keywords: Disconvergent fuzzy sequences, Divergent fuzzy sequences

1. Introduction

In a metric space sequence is an important tool to study many properties. The closure of a set A can be characterized using convergent sequences in A. The continuity of a function from one metric space to another can be characterized using convergent sequences. In the year 1965 Lotfi A.Zadeh [2] introduced the concept of fuzzy sets and fuzzy logic.In the year 1968 C.L.Chang [1] introduced Fuzzy topological spaces. In the year 2014 We [3] introduced fuzzy sequences in a metric space.

Let X be a non empty set. A fuzzy set A on N x X is called a fuzzy sequence in X. i.e., A: N x X —> [0, 1] is called a fuzzy sequence in X.
Let \((M,d)\) be a metric space and let \(A\) be a fuzzy sequence on \(M\). Let \(\alpha \in (0, 1]\). Let \(a \in M\). \(A\) is said to converge to \(a\) at level \(\alpha\) if 1. For each \(n \in \mathbb{N}\), there exists at least one \(x\) in \(M\) where \(A(n, x) \geq \alpha\) and 2. Given \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(d(x, a) < \varepsilon\) for all \(n \geq n_0\) and for all \(x\) with \(A(n, x) \geq \alpha\). i.e., given \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(n \geq n_0\) and \(A(n, x) \geq \alpha\) implies \(d(x, a) < \varepsilon\).

We write \(A \rightarrow a\).

In the year 2014 We [4] introduced fuzzy nets in a topological space and studied the properties of convergence.

In the same year 2014 We [5] introduced the concept of fuzzification of filters in topological space and studied the properties.

In the same year 2014 We [6] introduced the concept of fuzzy subsequences and limit points.

In this paper we introduce disconvergent fuzzy sequence in a metric space and also we introduce divergent fuzzy sequence in \(R\).

We have studied the behaviour of fuzzy sequences in any metric space \(M\). Hence all the definitions and results are true in \(R\) with usual metric. Since \(R\) has order in it. We can define and study fuzzy divergent sequence in \(R\).

2. Disconvergent Fuzzy Sequence

Definition 1:

A fuzzy sequence \(A\) in a metric space \(M\) is said to be a disconvergent fuzzy sequence at level \(\alpha\) if \(A\) is not convergent in \(M\) at level \(\alpha\).

Example 1:

Consider \(R\) with usual metric. Consider the fuzzy sequence given by \(A(n, x) = 1\) for all \(n\), for all \(x\). Take \(\alpha \in (0, 1]\). Claim: \(A\) is not convergent at level \(\alpha\). Suppose \(A\) converges to \(l\) at level \(\alpha\). Let \(\varepsilon > 0\) be given. Then there exists \(n_0 \in \mathbb{N}\) such that \(n \geq n_0\) and \(A(n, x) \geq \alpha\) implies \(d(x, l) < \varepsilon\). Take any \(n \geq n_0\). Now since \(A(n, x) = 1\) for all \(x\), \(A(n, x) \geq \alpha\) for all \(x\). Hence \(d(x, l) < \varepsilon\) for all \(x\), i.e., \(|x - l| < \varepsilon\) for all \(x \in R\). This is not true for \(x > l+2\varepsilon\). Our assumption is wrong. Hence \(A\) is not convergent at level \(\alpha\). Hence \(A\) is a disconvergent fuzzy sequence.

Note 1:

The disconvergence of a fuzzy sequence in a metric space can be classified to two different cases.

Definition 2: Disconvergence of first kind.
Let A be a disconvergent fuzzy sequence at level $\alpha$. If either the odd subsequence is disconvergent at level $\alpha$ or the even subsequence is disconvergent at level $\alpha$ then A is said to have disconvergence of first kind.

**Definition 3:** Disconvergence of second kind

Let A be a disconvergent fuzzy sequence at level $\alpha$. If the odd subsequence and the even subsequence of A converge to different limits at level $\alpha$ then A is said to have disconvergence of second kind.

**Example 2:** Consider $\mathbb{R}$ with usual metric. Consider the fuzzy sequence defined by $A(n, x) = 1$ for all $n$, for all $x$. Fix $\alpha$. The even subsequence $B$ is given by $B(n, x) = 1$ if $n$ is even and $B(n, x) = 0$ if $n$ is odd. Clearly $B$ is not convergent at level $\alpha$. Therefore A has disconvergence of first kind.

**Example 3:**
Consider $\mathbb{R}$ with usual metric. Consider the fuzzy sequence $A$ in $\mathbb{R}$ defined as

$$A(n, x) =\begin{cases} 1 & \text{if } n=2k \text{ and } x= 1/n, \\ 1 & \text{if } n=2k-1 \text{ and } x= n/(n+1), \\ 0 & \text{otherwise} \end{cases}$$

Take $\alpha \in (0, 1]$. The even subsequence $B$ is given by

$$B(n, x) =\begin{cases} 1 & \text{if } n=2k \text{ and } x= 1/n, \\ 0 & \text{otherwise} \end{cases}$$

The odd subsequence $C$ is given by

$$C(n, x) =\begin{cases} 1 & \text{if } n= 2k-1 \text{ and } x= n/(n+1), \\ 0 & \text{otherwise} \end{cases}$$

Clearly $B$ converges to 0 at level $\alpha$ and $C$ converges to 1 at level $\alpha$. Therefore A has disconvergence of second kind.

**3. Divergent fuzzy sequence in $\mathbb{R}$.

**Definition 4:**
Let A be a fuzzy sequence in $\mathbb{R}$ with usual metric. A is said to diverge to $+\infty$ at level $\alpha \in (0, 1]$ if for any given $k > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$
1. \( A(n, x) \geq \alpha \) for atleast one \( x \) in \( R \) and
2. \( A(n, x) \geq \alpha \) implies \( x > k \). We write \( A \rightarrow +\infty \)

**Definition 5:**
Let \( A \) be a fuzzy sequence in \( R \) with usual metric. \( A \) is said to diverge to \( -\infty \) at level \( \alpha \in (0, 1] \) if for any given \( k < 0 \), there exists \( n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \)

1. \( A(n, x) \geq \alpha \) for atleast one \( x \) in \( R \) and
2. \( A(n, x) \geq \alpha \) implies \( x < k \). We write \( A \rightarrow -\infty \)

**Definition 6:**
Let \( A \) be a fuzzy sequence in \( R \) with usual metric. \( A \) is said to be a divergent series at level \( \alpha \) if either \( A \) diverges to \( +\infty \) at level \( \alpha \) or \( A \) diverges to \( -\infty \) at level \( \alpha \).

**Example 4:**
Consider the fuzzy sequence \( A \) defined as \( A(n, x) = 1 \) if \( x = n \) and 0 otherwise. Take \( \alpha \in (0, 1] \). We claim that \( A \rightarrow +\infty \). Let \( K > 0 \) be given. Choose \( n_0 \in \mathbb{N} \) such that \( n_0 > k \). Let \( n \in \mathbb{N} \) and \( n \geq n_0 \) for this \( n \), \( A(n, n) = 1 \). Therefore \( A(n, x) \geq \alpha \) for atleast one \( x \) in \( R \). Now \( A(n, x) \geq \alpha \) implies \( A(n, n) = 1 \). This gives \( x = n \). Now \( x = n, n \geq n_0, n_0 > k \). Therefore \( A(n, n) \geq \alpha \) implies \( x = n \). Hence \( A \rightarrow +\infty \).

**Theorem 1:**
The concept of fuzzy divergent sequences in \( R \) is an extension of the concept of crisp divergent sequences in \( R \).

**Proof:** Let \( f \) be a crisp divergent sequence in \( R \). Consider the corresponding fuzzy sequence \( A_f \). We have to prove that \( A_f \) is divergent.

Case: 1 Let \( f \) diverge to \( +\infty \). Let \( k > 0 \) be given. Since \( f \) diverges to \( +\infty \), there exists \( n_0 \in \mathbb{N} \) such that \( f(n) = x_n > k \) for all \( n \geq n_0 \). Take \( \alpha \in (0, 1] \). Now let \( n \geq n_0 \). Then \( A_f(n, x_n) = 1 \). Hence \( A_f(n, x) \geq \alpha \) for atleast one \( x \) in \( R \). \( A_f(n, x) \geq \alpha \) implies \( x = x_n \) and hence \( x > k \). Hence given \( k > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \)

1. \( A_f(n, x) \geq \alpha \) for atleast one \( x \) in \( R \).
2. \( A_f(n, x) \geq \alpha \) implies \( x > k \).

Hence \( A_f \rightarrow +\infty \).

Case: 2 Let \( f \) diverge to \( -\infty \). As in case 1, \( A_f \rightarrow -\infty \).

Hence crisp sequence \( f \) diverges implies the fuzzy sequence \( A_f \) diverges. Hence the theorem.

**Result 1:** converse is also true.
Theorem 2:
Let f be a crisp sequence in R and let $A_f$ be the corresponding fuzzy sequence. If $A_f$ diverges then f also diverges.

Proof: Case 1: Let $A_f \longrightarrow +\infty$. Let $k > 0$ be given. Since $A_f$ diverges to $+\infty$, there exists $n_0 \in N$ such that for each $n \geq n_0$

1. $A_f(n,x) \geq \alpha$ for atleast one $x$ in R
2. $A_f(n,x) \geq \alpha$ implies $x > k$.

Now take any $n \in N$ with $n \geq n_0$. Let $f(n) = x_n$. Now $A_f(n,x_n) = 1$. Hence $A_f(n,x_n) \geq \alpha$. Therefore $x_n > k$.

Hence given any $k > 0$, there exists $n_0 \in N$ such that $f(n) = x_n > k$ for all $n \geq n_0$.

Hence f diverges to $+\infty$.

Case 2: Let $A_f \longrightarrow -\infty$ As in case 1, $f \longrightarrow -\infty$.

From case 1, case 2, we have $A_f$ diverges implies f diverges.

Theorem 3:
Let $(a_n)$ and $(b_n)$ be two crisp sequences in R both diverging to $+\infty$. Let A be the fuzzy sequence defined as $A(n,x) = 1$ if $x = a_n$ or $x = b_n$ and $A(n,x) = 0$ otherwise. Then A diverges to $+\infty$ at any level $\alpha > 0$.

Proof: $(a_n)$ diverges to $+\infty$ and $(b_n)$ diverges to $+\infty$.

1. By definition of A, for each $n \in N$, we have $a_n$ such that $A(n,a_n) = 1$. i.e., $A(n,a_n) \geq \alpha$.

2. Let $k > 0$ be given. Since $(a_n)$ diverges to $+\infty$, there exists $n_1 \in N$, such that $a_n > k$ for all $n \geq n_1$. Since $(b_n)$ diverges to $+\infty$, there exists $n_2 \in N$ such that $b_n > k$ for all $n \geq n_2$.

Let $n_0 = \max \{n_1, n_2\}$. Now let $n \geq n_0$ and $A(n,x) \geq \alpha$. Since $\alpha > 0$, $A(n,x) > \alpha$ implies $A(n,x) = 1$ and hence $x = a_n$ or $b_n$. Since $n \geq n_1, a_n > k$. Since $n \geq n_2, b_n > k$. Hence $x > k$. Therefore given $k > 0$, there exists $n_0 \in N$ such that $n \geq n_0$ and $A(n,x) \geq \alpha$ implies $x > k$. Hence A diverges to $+\infty$ at level $\alpha$.

Theorem 4:
Let $(a_n)$ and $(b_n)$ be two crisp sequences in R. Let A be the fuzzy sequence in R defined as $A(n,x) = 1$ if $x = a_n$ or $b_n$ and $A(n,x) = 0$ otherwise. If A diverges to $+\infty$ at some level $\alpha > 0$ then both $(a_n)$ and $(b_n)$ diverges to $+\infty$.

Proof: $A(n,x) = 1$ if $x = a_n$ or $b_n$ and 0 otherwise A diverges to $+\infty$. Claim: $(a_n)$ diverges to $+\infty$. Let $k > 0$ be given. Since A diverges to $+\infty$, there exists $n_0 \in N$ such that $n \geq n_0$ and $A(n,x) \geq \alpha$ implies $x > k$. Take $n \geq n_0$. A $(n,a_n) = 1 \geq \alpha$.

Hence $a_n > k$. Hence given $k > 0$, there exists $n_0 \in N$ such that $a_n > k$ for all $n \geq n_0$. 


Therefore \((a_n)\) diverges to \(+\infty\). Similarly \((b_n)\) diverges to \(+\infty\). Hence the theorem.

**Theorem 5:**
Let \((a_n)\) and \((b_n)\) be two crisp sequences in \(\mathbb{R}\). Let \(A\) be the fuzzy sequence defined as \(A(n, x) = 1\) if \(x = a_n\) or \(b_n\) and \(A(n, x) = 0\) otherwise. Then \(A\) diverges to \(+\infty\) iff both \((a_n)\) and \((b_n)\) diverge to \(+\infty\).

**Proof:** Follows from previous theorems.

**References**


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