Disconvergent and Divergent Fuzzy Sequences

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Abstract

We introduce the new concept disconvergent fuzzy sequences in a metric space. Also we define divergent fuzzy sequence in R .

Keywords: Disconvergent fuzzy sequences, Divergent fuzzy sequences

1. Introduction

In a metric space sequence is an important tool to study many properties. The closure of a set A can be characterized using convergent sequences in A . The continuity of a function from one metric space to another can be characterized using convergent sequences. In the year 1965 Lotfi A.Zadeh [2] introduced the concept of fuzzy sets and fuzzy logic. In the year 1968 C.L.Chang [1] introduced Fuzzy topological spaces. In the year 2014 We [3] introduced fuzzy sequences in a metric space.

Let X be a non empty set. A fuzzy set A on $N \times X$ is called a fuzzy sequence in X . i.e., $A: N \times X \rightarrow [0, 1]$ is called a fuzzy sequence in X .

Let (M, d) be a metric space and let A be a fuzzy sequence on M . Let $\alpha \in (0, 1]$. Let $a \in M$. A is said to converge to a at level α if 1. For each $n \in \mathbb{N}$, there exists atleast one x in M where $A(n, x) \geq \alpha$ and 2. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x, a) < \varepsilon$ for all $n \geq n_0$ and for all x with $A(n, x) \geq \alpha$. i.e., given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $A(n, x) \geq \alpha$ implies $d(x, a) < \varepsilon$.

We write $A \rightarrow a$.

In the year 2014 We [4] introduced fuzzy nets in a topological space and studied the properties of convergence.

In the same year 2014 We [5] introduced the concept of fuzzification of filters in topological space and studied the properties.

In the same year 2014 We [6] introduced the concept of fuzzy subsequences and limit points.

In this paper we introduce disconvergent fuzzy sequence in a metric space and also we introduce divergent fuzzy sequence in $\mathbb{R}$.

We have studied the behaviour of fuzzy sequences in any metric space M . Hence all the definitions and results are true in $\mathbb{R}$ with usual metric. Since $\mathbb{R}$ has order in it. We can define and study fuzzy divergent sequence in $\mathbb{R}$.

2. Disconvergent Fuzzy Sequence

Definition 1:

A fuzzy sequence A in a metric space M is said to be a disconvergent fuzzy sequence at level α if A is not convergent in M at level α .

Example 1:

Consider $\mathbb{R}$ with usual metric. Consider the fuzzy sequence given by $A(n, x) = 1$ for all n , for all x . Take $\alpha \in (0, 1]$. Claim: A is not convergent at level α . Suppose A converges to l at level α . Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $A(n, x) \geq \alpha$ implies $d(x, l) < \varepsilon$. Take any $n \geq n_0$. Now since $A(n, x) = 1$ for all x , $A(n, x) \geq \alpha$ for all x . Hence $d(x, l) < \varepsilon$ for all x . i.e., $|x - l| < \varepsilon$ for all $x \in \mathbb{R}$. This is not true for $x > l + 2\varepsilon$. Our assumption is wrong. Hence A is not convergent at level α . Hence A is a disconvergent fuzzy sequence.

Note 1:

The disconvergence of a fuzzy sequence in a metric space can be classified to two different cases.

Definition 2: Disconvergence of first kind.

Let A be a disconvergent fuzzy sequence at level α . If either the odd subsequence is disconvergent at level α or the even subsequence is disconvergent at level α then A is said to have disconvergence of first kind.

Definition 3: Disconvergence of second kind Let A be a disconvergent fuzzy sequence at level α . If the odd subsequence and the even subsequence of A converge to different limits at level α then A is said to have disconvergence of second kind.

Example 2: Consider R with usual metric. Consider the fuzzy sequence defined by $A(n, x) = 1$ for all n , for all x . Fix α . The even subsequence B is given by $B(n, x) = 1$ if n is even and $B(n, x) = 0$ if n is odd. Clearly B is not convergent at level α . Therefore A has disconvergence of first kind.

Example 3:

Consider R with usual metric. Consider the fuzzy sequence A in R defined as

$$A(n, x) = \begin{cases} 1 & \text{if } n=2k \text{ and } x=1/n. \\ 1 & \text{if } n=2k-1 \text{ and } x=n/(n+1) \\ 0 & \text{otherwise} \end{cases}$$

Take $\alpha \in (0, 1]$. The even subsequence B is given by

$$B(n, x) = \begin{cases} 1 & \text{if } n=2k \text{ and } x=1/n \\ 0 & \text{otherwise} \end{cases}$$

The odd subsequence C is given by

$$C(n, x) = \begin{cases} 1 & \text{if } n=2k-1 \text{ and } x=n/(n+1) \\ 0 & \text{otherwise} \end{cases}$$

Clearly B converges to 0 at level α and C converges to 1 at level α . Therefore A has disconvergence of second kind.

3. Divergent fuzzy sequence in R .

Definition 4:

Let A be a fuzzy sequence in R with usual metric. A is said to diverge to $+\infty$ at level $\alpha \in (0, 1]$ if for any given $k > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

1. $A(n, x) \geq \alpha$ for atleast one x in R and
2. $A(n, x) \geq \alpha$ implies $x > k$. We write $A \longrightarrow +\infty$

Definition 5:

Let A be a fuzzy sequence in R with usual metric. A is said to diverge to $-\infty$ at level $\alpha \in (0, 1]$ if for any given $k < 0$, there exists $n_0 \in N$ such that for each $n \geq n_0$

1. $A(n, x) \geq \alpha$ for atleast one x in R and
2. $A(n, x) \geq \alpha$ implies $x < k$. We write $A \longrightarrow -\infty$

Definition 6:

Let A be a fuzzy sequence in R with usual metric. A is said to be a divergent series at level α if either A diverges to $+\infty$ at level α or A diverges to $-\infty$ at level α .

Example 4:

Consider the fuzzy sequence A defined as $A(n, x) = 1$ if $x = n$ and 0 otherwise.

Take $\alpha \in (0, 1]$. We claim that $A \longrightarrow +\infty$. Let $K > 0$ be given. Choose $n_0 \in N$ such that $n_0 > k$. Let $n \in N$ and $n \geq n_0$ for this n , $A(n, n) = 1$. Therefore $A(n, x) \geq \alpha$ for atleast one x in R . Now $A(n, x) \geq \alpha$ implies $A(n, x) = 1$. This gives $x = n$. Now $x = n$, $n \geq n_0$, $n_0 > k$. Therefore $A(n, x) \geq \alpha$ implies $x > k$. Hence $A \longrightarrow +\infty$.

Theorem 1:

The concept of fuzzy divergent sequences in R is an extension of the concept of crisp divergent sequences in R .

Proof: Let f be a crisp divergent sequence in R . consider the corresponding fuzzy sequence A_f . We have to prove that A_f is divergent.

Case: 1 Let f diverge to $+\infty$. Let $k > 0$ be given. Since f diverges to $+\infty$, there exists $n_0 \in N$ such that $f(n) = x_n > k$ for all $n \geq n_0$. Take $\alpha \in (0, 1]$. Now let $n \geq n_0$. Then $A_f(n, x_n) = 1$. Hence $A_f(n, x) \geq \alpha$ for atleast one x in R . $A_f(n, x) \geq \alpha$ implies $x = x_n$ and hence $x > k$. Hence given $k > 0$, there exists $n_0 \in N$ such that for each $n \geq n_0$.

1. $A_f(n, x) \geq \alpha$ for atleast one x in R .
2. $A_f(n, x) \geq \alpha$ implies $x > k$.

Hence $A_f \longrightarrow +\infty$.

Case: 2 Let f diverge to $-\infty$. As in case 1, $A_f \longrightarrow -\infty$

Hence crisp sequence f diverges implies the fuzzy sequence A_f diverges. Hence the theorem.

Result 1: converse is also true.

Theorem 2:

Let f be a crisp sequence in $\mathbb{R}$ and let A_f be the corresponding fuzzy sequence. If A_f diverges then f also diverges.

Proof: Case 1: Let $A_f \rightarrow +\infty$. Let $k > 0$ be given. Since A_f diverges to $+\infty$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

1. $A_f(n, x) \geq \alpha$ for atleast one x in $\mathbb{R}$
2. $A_f(n, x) \geq \alpha$ implies $x > k$.

Now take any $n \in \mathbb{N}$ with $n \geq n_0$. Let $f(n) = x_n$. Now $A_f(n, x_n) = 1$. Hence $A_f(n, x_n) \geq \alpha$. Therefore $x_n > k$.

Hence given any $k > 0$, there exists $n_0 \in \mathbb{N}$ such that $f(n) = x_n > k$ for all $n \geq n_0$. Hence f diverges to $+\infty$.

Case: 2 Let $A_f \rightarrow -\infty$ As in case 1, $f \rightarrow -\infty$

From case 1, case 2, we have A_f diverges implies f diverges.

Theorem 3:

Let (a_n) and (b_n) be two crisp sequences in $\mathbb{R}$ both diverging to $+\infty$. Let A be the fuzzy sequence defined as $A(n, x) = 1$ if $x = a_n$ or $x = b_n$ and $A(n, x) = 0$ otherwise. Then A diverges to $+\infty$ at any level $\alpha > 0$.

Proof: (a_n) diverges to $+\infty$ and (b_n) diverges to $+\infty$.

1. By definition of A , for each $n \in \mathbb{N}$, we have a_n such that $A(n, a_n) = 1$. i.e., $A(n, a_n) \geq \alpha$.

2. Let $k > 0$ be given. Since (a_n) diverges to $+\infty$, there exists $n_1 \in \mathbb{N}$, such that $a_n > k$ for all $n \geq n_1$. Since (b_n) diverges to $+\infty$, there exists $n_2 \in \mathbb{N}$ such that $b_n > k$ for all $n \geq n_2$.

Let $n_0 = \max\{n_1, n_2\}$. Now let $n \geq n_0$ and $A(n, x) \geq \alpha$. Since $\alpha > 0$, $A(n, x) > \alpha$ implies $A(n, x) = 1$ and hence $x = a_n$ or b_n . Since $n \geq n_1$, $a_n > k$. Since $n \geq n_2$, $b_n > k$. Hence $x > k$. Therefore given $k > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $A(n, x) \geq \alpha$ implies $x > k$. Hence A diverges to $+\infty$ at level α .

Theorem 4:

Let (a_n) and (b_n) be two crisp sequences in $\mathbb{R}$. Let A be the fuzzy sequence in $\mathbb{R}$ defined as $A(n, x) = 1$ if $x = a_n$ or b_n and $A(n, x) = 0$ otherwise. If A diverges to $+\infty$ at some level $\alpha > 0$ then both (a_n) and (b_n) diverges to $+\infty$.

Proof: $A(n, x) = 1$ if $x = a_n$ or b_n and 0 otherwise A diverges to $+\infty$. Claim: (a_n) diverges to $+\infty$. Let $k > 0$ be given. Since A diverges to $+\infty$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $A(n, x) \geq \alpha$ implies $x > k$. Take $n \geq n_0$, $A(n, a_n) = 1 \geq \alpha$. Hence $a_n > k$. Hence given $k > 0$, there exists $n_0 \in \mathbb{N}$ such that $a_n > k$ for all $n \geq n_0$.

Therefore (a_n) diverges to $+\infty$. Similarly (b_n) diverges to $+\infty$. Hence the theorem.

Theorem 5:

Let (a_n) and (b_n) be two crisp sequences in $\mathbb{R}$. Let A be the fuzzy sequence defined as $A(n, x) = 1$ if $x = a_n$ or b_n and $A(n, x) = 0$ otherwise. Then A diverges to $+\infty$ iff both (a_n) and (b_n) diverge to $+\infty$.

Proof: Follows from previous theorems.

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