The q -Numerical Range of a Nilpotent 4×4 Matrix

Yu Ogasawara

Graduate School of Hirosaki University
Hirosaki 036-8561, Japan

Hiroshi Nakazato

Department of Mathematical Sciences
Faculty of Science and Technology
Hirosaki University, Hirosaki 036-8561, Japan

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Abstract

In this note the q -numerical range $W_q(A)$ of a matrix A is introduced and the algorithm to compute the boundary of $W_q(A)$ is provided. Especially the equation of the boundary of $W_q(A)$ is provided for a 4×4 nilpotent matrix A and some real number $0 < q < 1$.

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1 Introduction

Let A be an $n \times n$ complex matrix and $0 \leq q \leq 1$. The q -numerical range of A is defined and denoted by

$$W_q(A) = \{\eta^* A \xi : \xi, \eta \in \mathbb{C}^n, \xi^* \xi = \eta^* \eta = 1, \eta^* \xi = q\}. \quad (1.1)$$

This set satisfies $W_q(A \otimes I_m) = W_q(A)$ and this property is useful to analyze the q -maximal fidelity of quantum operations. If $q = 1$, the range $W_q(A)$ is reduced to the classical numerical range

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

In 1919, Hausdorff [4] proved the convexity of the range $W(A)$. In 1984, Tsing [8] showed the following formula

$$W_q(A) = \{q\xi^* A \xi + \sqrt{1 - q^2} w \sqrt{\xi^* A^* A \xi - \xi^* A \xi} : w \in \mathbb{C}, |w| \leq 1, \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

We can prove that the function

$$\phi(z) = \max\{\sqrt{\xi^* A^* A \xi - \xi^* A \xi} : \xi \in \mathbb{C}^n, \xi^* \xi = 1, \xi^* A \xi = z\}$$

is concave on $W(A)$. By using these properties, Tsing proved the convexity of $W_q(A)$. It follows from the Tarski-Seidenberg theorem that the boundary $\partial W_q(A)$ of the q -numerical range lies on an algebraic curve. C. K. Li [6] provides a Matlab program to plot $W_q(A)$ numerically (cf. [7]). A performable algorithm to generate the polynomial $g(x, y)$ for which

$$\{(x, y) \in \mathbb{R}^2 : x + iy \in \partial W_q(A)\} \subset \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\},$$

$$W_q(A) = \text{Conv}(\{x + iy : (x, y) \in \mathbb{R}^2, g(x, y) = 0\})$$

is given in [1], [5] (cf. [3]). We introduce a compact convex set $\Gamma(A)$ by

$$\Gamma(A) = \{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 : x_1 + ix_2 \in W(A), u_1^2 + u_2^2 \leq \phi(x_1 + ix_2)^2\}.$$

Define an orthogonal projection Π_q of $\mathbb{R}^4$ onto $\mathbb{C} \cong \mathbb{R}^2$ by

$$\Pi_q(x_1, x_2, u_1, u_2) = (qx_1 + \sqrt{1 - q^2}u_1) + i(qx_2 + \sqrt{1 - q^2}u_2).$$

Then Tsing's formula is rewritten as

$$W_q(A) = \Pi_q(\Gamma(A)).$$

This formula provides a principle to compute the equation $g(x, y) = 0$ of the boundary $W_q(A)$. The q -numerical range of some typical 3×3 matrices are given in [2]. It is rather hard to compute the polynomial $g(x, y)$ for a generic unitarily irreducible 4×4 matrix A by using a standard personal computer. As a first step to treat a generic 4×4 matrix, we treat the following 4×4 nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.2)$$

We easily find that the above matrix N and the following 3×3 nilpotent matrix N_1 have many similar properties:

$$N_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as objects for the matrix analysis. In the paper [1], the equation of the boundary of $W_q(N_1)$ is given.

2 The Davis-Wielandt shell

The standard method to generate the function $\phi = \phi_A$ on the numerical range $W(A)$ for an $n \times n$ matrix is given by the formula

$$\phi_A(z) = \sqrt{h(z) - |z|^2},$$

$$h(z) = \max\{s : (z, s) \in W(A, A^*A)\}$$

where

$$W(A, A^*A) = \{(z, s) \in \mathbb{C} \times \mathbb{R} : z = \xi^* A \xi, s = \xi^* A^* A \xi, \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

We shall generate a real polynomial $L_0(X, Y, Z)$ for which $L_0(X, Y, Z) = 0$ for a generic point $(X + iY, Z)$ of the boundary of $W(A, A^*A)$. As it is mentioned in [1], the algebraic surface $L_0(X, Y, Z) = 0$ is characterized as the dual surface of the algebraic surface $G_A(x, y, z, 1) = 0$ defined by

$$G_A(x, y, z, t) = \det(x\Re(A) + y\Im(A) + zA^*A + tI_n),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$. By using Sylvester's resultant, we can compute the polynomials G_N and L_{0N} for the nilpotent N defined by (1.2).

Theorem 2.1 *Suppose that N is the 4×4 nilpotent matrix given by (1.2). Then the polynomials G_N and L_{0N} are given by the following:*

$$\begin{aligned} 16G_N(x, y, z, 1) = & -3x^4 - 2x^2y^2 + y^4 + 8x^3z + 8xy^2z - 4x^2z^2 - 4y^2z^2 \\ & + 16x^3 + 16xy^2 - 8x^2z - 24y^2z - 32xz^2 + 16z^3 \\ & - 24x^2 - 24y^2 - 64xz + 80z^2 + 96z + 16, \end{aligned}$$

$$\begin{aligned} L_{0N}(X, Y, Z) = & 196X^6 + 504X^4Y^2 + 436X^2Y^4 + 128Y^6 + 448X^5Z \\ & + 800X^3Y^2Z + 352XY^4Z + 536X^4Z^2 + 792X^2Y^2Z^2 + 256Y^4Z^2 \end{aligned}$$

$$\begin{aligned}
& +336X^3Z^3 + 336XY^2Z^3 + 164X^2Z^4 + 160Y^2Z^4 + 16XZ^5 + 32Z^6 \\
& +84X^5 + 136X^3Y^2 + 52XY^4 - 184X^4Z - 376X^2Y^2Z - 208Y^4Z \\
& -520X^3Z^2 - 42XY^2Z^2 - 624X^2Z^3 - 344Y^2Z^3 - 332XZ^4 - 15Z^5 \\
& +37X^4 + 50X^2Y^2 + 13Y^4 - 64X^3Z - 32XY^2Z + 10X^2Z^2 + 102Y^2Z^2 \\
& +72XZ^3 + 89Z^4 + 8X^3 + 8XY^2 - 38X^2Z - 14Y^2Z - 20XZ^2 - 22Z^3 \\
& +X^2 + Y^2 - 8XZ + Z^2 - Z.
\end{aligned}$$

The above degree 6 polynomial L_{0N} has 48 terms.

3 The boundary of the q -numerical range

By using the equation $L_{0A}(X, Y, Z) = 0$ of the boundary of the Davis-Wielandt shell $W(A, A^*A)$, the equation of the boundary of $\Gamma(A)$ is given by

$$L_{0,A}(x_1, x_2, x_1^2 + x_2^2 + u_1^2 + u_2^2) = 0.$$

We consider the orthogonal projection Π_q of $\mathbb{R}^4$ onto the plane $\mathbb{C} \cong \mathbb{R}^2$. The algorithm to compute the equation of the boundary of $W_q(A)$ is given by the following. We substitute

$$x_1 = \frac{1}{q}(x - \sqrt{1 - q^2}u_1), \quad x_2 = \frac{1}{q}(y - \sqrt{1 - q^2}u_2)$$

into the polynomial

$$L(x_1, x_2, u_1, u_2) = L_{0,A}(x_1, x_2, x_1^2 + x_2^2 + u_1^2 + u_2^2).$$

The polynomial $g(x, y)$ vanishing on the boundary of $W_q(A)$ is obtained by the successive eliminations of u_1, u_2 from the equations

$$M(x, y, u_1, u_2) = L(1/q(x - \sqrt{1 - q^2}u_1), 1/q(y - \sqrt{1 - q^2}u_2), u_1, u_2),$$

$$M_{u_1}(x, y, u_1, u_2) = 0, M_{u_2}(x, y, u_1, u_2) = 0.$$

Firstly we take a simple factor $m(x, y, u_2)$ of the resultant of M and M_{u_1} with respect to u_1 . Then the polynomial $g(x, y)$ appears as a simple factor of the resultant of m and m_{u_2} with respect to u_2 . This process essentially coincides with that in [1]. We provide the equation of the boundary of $W_q(N)$ for $q = 1599/1601$, $\sqrt{1 - q^2} = 80/1601$. We choose this value of q by using a Pythagorean triple $(1599, 80, 1601)$ for which $80/1601$ is rather close to 0.

Theorem 3.1 *Suppose that N is the 4×4 nilpotent matrix given by (1.2) and $q = 1599/1601$. Then every point $x+iy$ of the boundary of $W_q(N)$ ($(x, y) \in \mathbb{R}^2$) satisfies the equation $g(x, y) = 0$ for the following degree 20 polynomial with 121 terms in the expanded form:*

$$\begin{aligned} g(x, y) = & 2^{25} \cdot 1601^{20} (x^2 + y^2)^8 (2563201x^2 + 6400y^2)^2 \\ & + 2^{24} \cdot 1601^{19} \times 1599x(x^2 + y^2)^7 (2563201x^2 + 6400y^2)(12816005x^2 + 10259204y^2) \\ & + 3 \times 2^{22} \cdot 1601^{18}(x^2 + y^2)^6 (16097660987498563201x^6 + 77212976642472795738x^4 y^2 \\ & + 66742213824651891204 x^2 y^4 + 55431615638195200 y^6) \\ & + \dots + 3^2 \times 59 \times 139 \times 467 \times 1686403 \times 1599^{20} \end{aligned}$$

By using the above polynomial $g(x, y)$, we shall determine some characteristic invariants of $W_q(N)$ for $q = 1599/1601$. We determine the least rectangle R containing $W_{1599/1601}(N)$ with edges parallel to the real and imaginary axes. Since N is a real matrix, the range $W_q(N)$ is symmetric with respect to the real axis. So the values

$$\max\{\Re(z) : z \in W_q(A)\}, \quad \min\{\Re(z) : z \in W_q(A)\}$$

are attained at the points on the line $\Im(z) = 0$. We have

$$\begin{aligned} g(x, 0) = & (1077774652541338214464 x^6 - 1616998573019546777728x^5 \\ & - 1483284369114705689488x^4 + 1074246192203940172736x^3 \\ & + 1340498694635581184028x^2 + 501912410701368463992x \\ & + 75110179897565403201) \cdot (2155549305082676428928x^6 \\ & + 4310425421750273945792x^5 + 3761107984965466060624x^4 \\ & + 2143110240926924646272x^3 + 936168969993563648160x^2 \\ & + 266550351920353708740x + 33073219699220361609) \\ & (1077774652541338214464x^4 + 1076428275711180390336x^3 \\ & + 536199614169005977632x^2 + 133882445823897651192x \\ & + 16714399662118390401)^2. \end{aligned}$$

The repeated quartic factor of $g(x, 0)$ is positive definite on the real line. The maximum of $\Re(z)$ for $z \in W_{1599/1601}(N)$ is attained by the maximal real root of the factor $1077774652541338214464 x^6 + \dots$. The maximum is approximately 1.55530380705. The minimum of $\Re(z)$ for $z \in W_{1599/1601}(N)$ is attained

by the minimal real root of the factor $2155549305082676428928x^6 + \dots$. The minimum is approximately -0.560765773519 . The values

$$\max\{\Im(z) : z \in W_q(A)\}, \quad \min\{\Im(z) : z \in W_q(A)\}$$

are roots of the simple factors of the resultant of $g(x, y)$ and $g_x(x, y)$ with respect to x . The two simple factors of the resultant are given by

$$\begin{aligned} p_\epsilon(y) = & 8622197220330705715712y^6 + \epsilon 15088845135578735002496y^5 \\ & + 1320307598271894272080y^4 + \epsilon 2698473238964062310560y^3 \\ & + 2273815199350434636936y^2 + \epsilon 804299421253508966448y \\ & + 82442364732689699205. \end{aligned}$$

($\epsilon = \pm 1$). The maximum of $\Im(z)$ for $z \in W_{1599/1601}(N)$ is approximately 0.233123492872 . In Figure 1, we provide a graphic of the curve $g(x, y) = 0$. The outer arc of this figure represents the boundary $W_{1599/1601}(N)$.

Figure1: $\partial W_q(N)$ and its related envelope curve

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