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Analytical Treatment for Solving System of Fuzzy IVPs Using Residual Power Series Approach

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Abstract

The aim of the present analysis is present a relatively new analytical treatment, called residual power series (RPS) method, for solving system of fuzzy initial value problems under strongly generalized differentiability. The technique methodology provides the solution in the form of a rapidly convergent series with easily computable components using symbolic computation software. Several computational experiments are given to show the good performance and potentiality of the proposed procedure. The results reveal that the present simulated method is very effective, straightforward and powerful methodology to solve such fuzzy equations.

Keywords: Fuzzy differential equations, Residual power series method, Initial value problems, Strongly generalized differentiability

1. Introduction

Fuzzy differential equations (FDEs) are extensively used in modeling of complex phenomena arising in applied mathematics, physics, and engineering, including fuzzy control theory, quantum optics, atmosphere, measure theory and dynamical systems [1-6]. In many cases, data about these physical phenomena is pervaded under uncertainty, which may arise in the experiment part, data collection, measurement process as well as when determining the initial values. Most of the uncertain practical problems under the differential sense require the solutions of the corresponding FDEs which satisfy the given fuzzy initial conditions; therefore, fuzzy problems should be solved.

In general, there exists no method that yields an explicit solution for FDEs due to the complexities of uncertain parameters involving these equations. Anyhow, in most cases, analytical solutions cannot be found, where the solutions of such equations are always in demand due to practical interests. Therefore, efficient reliable computer stimulation is required. To deal with this in more realistic situations, FDEs are commonly solved approximately using numerical techniques.

In this paper, we discuss and provide numerical approximate solutions for system of FIVP of the form

$$\begin{aligned} x_1'(t) &= f_1\big(t, x_1(t), x_2(t), \dots, x_n(t)\big), \\ x_2'(t) &= f_2\big(t, x_1(t), x_2(t), \dots, x_n(t)\big), \\ &\vdots \\ x_n'(t) &= f_n\big(t, x_1(t), x_2(t), \dots, x_n(t)\big), \end{aligned} \tag{1}$$

subject to the initial conditions

$$x_1(t_0) = \alpha_1, x_2(t_0) = \alpha_2, \dots, x_n(t_0) = \alpha_n, \tag{2}$$

where $t \in [t_0, t_0 + a]$, t_0, α_i, a are real finite constants with $a > 0$, $f_i: [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}$ are linear or nonlinear continuous functions in term of $t, x_i, i = 1,2, \dots, n$, and $x_i(t), i = 1,2, \dots, n$, are unknown functions of independent variable t to be determined. If $f_i, i = 1,2, \dots, n$, be crisp functions, then the solution of the system is crisp. However, if $f_i, i = 1,2, \dots, n$, be fuzzy functions, then this system may only process fuzzy solutions. Throughout this analysis, we assume that $f_i, x_i, i = 1,2, \dots, n$, are analytic functions on the given interval. Also, we assume that $f_i, i = 1,2, \dots, n$, satisfies all the necessary requirements for the existence of a unique solution.

Series expansions are very important aids in numerical calculations, especially for quick estimates made in hand calculation. Solutions of the FDEs can often be expressed in terms of series expansions. However, the RPS technique is an analytical as well as numerical method for solving different types of ordinary and partial differential equations, integral equation and integro-differential equation [7-16]. The methodology is effective and easy to construct power series solution for strongly linear and nonlinear systems of FIVPs without linearization, perturbation, or discretization [17-23]. Different from the classical power series method, the RPS technique does not need to compare the coefficients of the corresponding terms and recursion relations are not required, which computes the coefficients of its power series by a chain of linear equations of n-variable, where n is number of equations in the given system. Other numerical methods can be found in [24-35].

This article is organized as follows. In the next section, we revisit briefly some necessary definitions and preliminary results from the fuzzy calculus theory including the strongly generalized differentiability. Formulation for solving the system of FIVPs is presented in Section 3. In Section 4, the RPS algorithm is built and introduced to illustrate the capability of proposed approach. Numerical experiments and simulation results are presented in Section 5.

2. Excerpts of Fuzzy Analysis Theory

The material in this section is basic in certain sense. For the reader's convenience, we present some necessary definitions and notations from fuzzy calculus theory which be used throughout the paper.

A fuzzy set u in $\mathbb{R}$ is characterized by its membership function $u: \mathbb{R} \to [0,1]$, while $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in \mathbb{R}$. A fuzzy set u on $\mathbb{R}$ is called convex, if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, we have $u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$; is called upper semicontinuous, if the set $\{s \in \mathbb{R} \mid u(s) > \alpha\}$ is closed for each $\alpha \in [0,1]$; and is called normal, if there is $s \in \mathbb{R}$ such that $u(s) = 1$.

Definition 2.1. [36] A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

For each $r \in (0,1]$, put $[u]^r = \{s \in \mathbb{R}: u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R}: u(s) > 0\}}$, where $\overline{\{\cdot\}}$ is the closure of $\{\cdot\}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is compact convex subset of $\mathbb{R}$ for each $r \in [0,1]$ and $[u]^1 \neq \phi$ [35]. Thus, if u is a fuzzy number, then $[u]^r = [\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r) = \min\{s: s \in [u]^r\}$ and $\overline{u}(r) = \max\{s: s \in [u]^r\}$ for each $r \in [0,1]$. Hence, the r -level set $[u]^r$ is a non-empty compact interval for each $r \in [0,1]$ and any $u \in \mathbb{R}_F$, where $\mathbb{R}_F$ denote the set of fuzzy numbers on $\mathbb{R}$. The symbol $[u]^r$ is called the r-cut representation or parametric form of a fuzzy number u. The previous descriptions leads to the following characterization theorem of fuzzy number in terms of the two endpoint functions $\underline{u}(r)$ and $\overline{u}(r)$.

Theorem 2.1. [37] Suppose that $\underline{u}: [0,1] \to \mathbb{R}$ and $\overline{u}: [0,1] \to \mathbb{R}$ satisfy the following conditions: First, $\underline{u}$ is a bounded increasing function and $\overline{u}$ is a bounded decreasing function with $\underline{u}(1) \leq \overline{u}(1)$; second, for each $\alpha \in (0,1]$, $\underline{u}$ and $\overline{u}$ are left-hand continuous functions at $\alpha = r$; third, $\underline{u}$ and $\overline{u}$ are right-hand continuous functions at $r = 0$. Then, $u: \mathbb{R} \to [0,1]$ defined by $u(s) = \sup\{r: \underline{u}(r) \leq s \leq \overline{u}(r)\}$ is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$. Furthermore, if $u: \mathbb{R} \to [0,1]$ is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$, then the functions $\underline{u}$ and $\overline{u}$ satisfy the conditions.

The metric structure on $\mathbb{R}_F$ is given by the Hausdorff distance $D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ such that $D(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ for arbitrary fuzzy numbers u and v. In [36], it has been proved that $(\mathbb{R}_F, D)$ is a complete metric space.

Definition 2.2. [36] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference (Hukuhara difference) of u and v, and is denoted by $u \ominus v$.

Here, it is worth mentioning that the sign "$\ominus$" stands always for H-difference and $u \ominus v \neq u + (-1)v$. Usually, we denote $u + (-1)v$ by $u - v$. If the H-difference $u \ominus v$ exists, then $[u \ominus v]^r = [\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)]$. It follows that the Hukuhara differentiable function has increasing length of support [37]. To avoid this difficulty, we consider the following fundamental definition with the property that the support has increasing or decreasing length.

Definition 2.3. [36] Let $x: [a, b] \to \mathbb{R}_F$ and $t_0 \in [a, b]$. We say that x is strongly generalized differentiable at t_0, if there exists an element $x'(t_0) \in \mathbb{R}_F$ such that either

i) for each $h > 0$ sufficiently close to 0, the H-differences $x(t_0 + h) \ominus x(t_0)$, $x(t_0) \ominus x(t_0 - h)$ exist and
$$\lim_{h \to 0^+} \frac{x(t_0 + h) \ominus x(t_0)}{h} = \lim_{h \to 0^+} \frac{x(t_0) \ominus x(t_0 - h)}{h} = x'(t_0),$$
ii) for each $h > 0$ sufficiently close to 0, the H-differences $x(t_0) \ominus x(t_0 + h)$, $x(t_0 - h) \ominus x(t_0)$ exist and
$$\lim_{h \to 0^+} \frac{x(t_0) \ominus x(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{x(t_0 - h) \ominus x(t_0)}{-h} = x'(t_0).$$

Here, the limit is taken in the metric space $(\mathbb{R}_F, D)$ and at the endpoints of $[a, b]$, we consider only one-sided derivatives. If x is differentiable at any point $t \in [a, b]$, then we say that x is differentiable on $[a, b]$. Furthermore, we say that x is (1)-differentiable on $[a, b]$, if x is differentiable in the sense of (i) and its derivative is denoted $D_1 x$, while x is (2)-differentiable on $[a, b]$, if x is differentiable in the sense of (ii) and its derivative is denoted $D_2 x$ (see [36]). The first condition of Definition 2.3 corresponds to the Hukuhara-derivative introduced in [38], so this differentiability-type is a generalization of the H-derivative. Frequently, we will write simply x_{1r} and x_{2r} instead of $\underline{x}(r)$ and $\overline{x}(r)$, respectively, for each $r \in [0,1]$.

Theorem 2.2. [39] Let $x: [a, b] \to \mathbb{R}_F$ and put $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ for each $r \in [0,1]$.

i) If x is (1)-differentiable, then x_{1r} and x_{2r} are differentiable functions and $[D_1 x(t)]^r = [\, x'_{1r}(t), x'_{2r}(t)]$,
ii) If x is (2)-differentiable, then x_{1r} and x_{2r} are differentiable functions and $[D_2 x(t)]^r = [x'_{2r}(t), x'_{1r}(t)]$.

The previous subsequent theorems show us a way to translate a fuzzy IVP into a system of crisp IVP without the need to consider the fuzzy setting approach. As a conclusion, one does not need to rewrite the numerical methods for the system of crisp IVPs in fuzzy setting, but instead, we can use the methods directly on the obtained system of crisp IVPs.

Definition 2.4. Let $x: [a, b] \to \mathbb{R}_F$. Then, we say that x is continuous at $t_0 \in [a, b]$ if for every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $D(x(t), x(t_0)) < \varepsilon$, for each $t \in [a, b]$, whenever $|t - t_0| < \delta$.

We say that x is continuous on $[a, b]$, if x is continuous at each $t_0 \in [a, b]$ such that the continuity is one-sided at endpoints of $[a, b]$, that is, x is continuous on $[a, b]$ if and only if x_{1r} and x_{2r} are continuous on $[a, b]$.

3. Formulation System of Fuzzy Initial Value Problems

In this section, we study the system of FIVPs under the concept of strongly generalized differentiability in which the fuzzy differential equation is converted into equivalent system of crisp system of IVPs for each type of differentiability. These can be done if the initial value is fuzzy number, the solution is fuzzy function, and consequently the derivative must be considered as fuzzy derivative. Furthermore, a computational algorithm is provided to guarantee the procedure and to confirm the performance of the proposed technique.

Prior to construct approximate possible fuzzy solutions of fuzzy system (1) and (2), we write the fuzzy functions $x_i(t), i = 1,2, \dots, n,$ in terms of its r-cut representation form to get that $[x_i(t)]^r = \left[x_{(i)1r}(t), x_{(i)2r}(t)\right]$ and $[x_i(t_0)]^r = \left[\alpha_{(i)1r}, \alpha_{(i)2r}\right], i = 1,2, \dots, n$. Thus, by considering the parametric form for both sides of system (1) and (2), one can write

$$[x_i(t)]^r = \left[f_i\big(t, x_k(t)\big)\right]^r, k = 1,2, \dots, n, \tag{3}$$

subject to the initial conditions

$$[x_i(t_0)]^r = [\alpha_i]^r, i = 1,2, \dots, n, \tag{4}$$

where the endpoint functions of $\left[f_i\big(t, x_k(t)\big)\right]^r$ are given by the set

$$\begin{aligned}\left[f_i\big(t, x_k(t)\big)\right]^r &= \left[f_{(i)1r}\big(t, x_k(t)\big), f_{(i)2r}\big(t, x_k(t)\big)\right] \\ &= \left[f_{(i)1r}\big(t, x_{(k)1r}(t), x_{(k)2r}(t)\big), f_{(i)2r}\big(t, x_{(k)1r}(t), x_{(k)2r}(t)\big)\right].\end{aligned}$$

Definition 3.1. [39] Let $x: [a, b] \to \mathbb{R}_F$ such that $D_1 x$ and $D_2 x$ exists. If x with $D_1 x$ satisfy the fuzzy system (1) and (2), then we say that x is a (1)-solution of FIVPs (1) and (2). Similarly, if x with $D_2 x$ satisfy the fuzzy system (1) and (2), then we say that x is a (2)-solution of FIVPs (1) and (2).

Let x be a (n)-solution, then by utilizing Theorems 2.2, we can thus translate the fuzzy system (1) and (2) into system of crisp DEs, hereafter, called the corresponding (n)-system. In some cases, we can't decompose the membership function of the fuzzy solution $[x(t)]^r$ as a function defined on $\mathbb{R}$ for each $t \in [a, b]$. Then, we can leave the (n)-solution in terms of its r-cut representation form.

4. Construction of the RPS Technique for Systems of FIVPs

The RPS technique consists in expressing the solutions of system of IVPs (1) and (2) as a power series expansion about the initial point $t = t_0$. To achieve our goal, we suppose that these solutions take the form

$$x_i(t) = \sum_{m=0}^{\infty} x_{i,m}(t), i = 1,2,\dots,n, \tag{5}$$

where $x_{i,m}$ are terms of approximations and are given as $x_{i,m}(t) = c_{i,m}(t - t_0)^m$.

Notice that in writing out the term corresponding to $m = 0$ in PS (5), we have adopted the convention that $(t - t_0)^0 = 1$ even when $t = t_0$. If $t = t_0$, then all terms of the PS (5) are vanishing for $m \geq 1$ and so. Anyhow, let $x_{i,0}(t)$ be the initial guesses approximations of $x_i(t), i = 1,2,\dots,n$. Then, since they satisfy the initial conditions (2), we have $x_{i,0}(t_0) = x_i(t_0) = \alpha_i, i = 1,2,\dots,n$. Thus, after choosing $x_{i,0}(t) = x_i(t_0)$ as initial guesses approximations of $x_i(t)$ and calculating $x_{i,m}(t)$ for $m = 1,2,\dots,k$, then the solutions $x_i(t)$ of system (1) and (2) can be approximated by the following kth-truncated series

$$x_i^k(t) = \sum_{m=0}^{k} c_{i,m}(t - t_0)^m, i = 1,2,\dots,n. \tag{6}$$

Prior to applying the RPS technique, we rewrite the system as

$$x_i'(t) - f_i\big(t, x_1(t), x_2(t),\dots,x_n(t)\big) = 0, i = 1,2,\dots,n. \tag{7}$$

The subsisting of kth-truncated series $x_i^k(t)$ into system (7) leads to the following definition of kth-residual functions

$$\begin{aligned}\mathrm{Res}_i^k(t) = &\sum_{m=1}^{k} m c_{i,m}(t - t_0)^{m-1} \\ &-f_i\left(t, \sum_{m=0}^{k} c_{1,m}(t - t_0)^m, \sum_{m=0}^{k} c_{2,m}(t - t_0)^m,\dots,\sum_{m=0}^{k} c_{n,m}(t - t_0)^m\right),\end{aligned} \tag{8}$$

in which the ∞th-residual functions is given by $\mathrm{Res}_i^\infty(t) = \lim_{k\to\infty} \mathrm{Res}_i^k(t), i = 1,2,\dots,n$.

Clear that $\mathrm{Res}_i^\infty(t) = 0$ for each $t \in [t_0, t_0 + a]$. This show that $\mathrm{Res}_i^\infty(t)$ are infinitely many times differentiable at $t = t_0$. On the other hand, $\frac{d^n}{dt^n}\mathrm{Res}_i^\infty(t_0) = \frac{d^n}{dt^n}\mathrm{Res}_i^k(t_0) = 0$, for each $n = 1,2,\dots,k$. To obtain the values of the coefficients $c_{i,m}$, we need to solve the following algebraic system $\frac{d^{(n-1)}}{dt^{(n-1)}}\mathrm{Res}_i^n(t_0) = 0, n = 1,2,\dots,k$.

To determine the first unknown coefficient, $c_{i,1}$, we defined the 1st-residual function as

$\text{Res}_i^1(t) = \sum_{m=1}^{1} mc_{i,m}(t-t_0)^{m-1} - f_i\big(t, \sum_{m=0}^{1} c_{1,m}(t-t_0)^m, \dots, \sum_{m=0}^{1} c_{n,m}(t-t_0)^m\big)$. Substituting $t = t_0$ into the 1st-residual function and then using the fact $\text{Res}_i^\infty(t_0) = \text{Res}_i^1(t_0) = 0$, we have $c_{i,1} = f_i(t_0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) = f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big), i = 1,2, \dots, n$. Thus, by using the 1st-truncated series, the 1st-approximate solutions can be written as

$$x_i^1(t) = x_i(t_0) + f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)(t-t_0), i = 1,2, \dots, n. \tag{9}$$

For second unknown coefficient, $c_{i,2}$, differentiate both sides of formula (8) with respect to t, put $k = 2$, and then substitute $t = t_0$ to get that

$$\frac{d}{dt}\text{Res}_i^2(t_0) = 2c_{i,2} - \frac{\partial}{\partial t} f_i(t_0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) - \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i\big(t_0, c_{1,0}, c_{2,0}, \dots, c_{n,0}\big), i = 1,2, \dots, n. \tag{10}$$

Similarly, by using the fact $\frac{d}{dt}\text{Res}_i^2(t_0) = \frac{d}{dt}\text{Res}_i^\infty(t_0) = 0$, we have

$$c_{i,2} = \frac{1}{2}\left[\frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)\right], i = 1,2, \dots, n. \tag{11}$$

Now, by using the 2nd-truncated series, the 2nd-approximate solutions can be written as

$$x_i^2(t) = x_i(t_0) + f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)(t-t_0) + \frac{1}{2}\left[\frac{\partial}{\partial t} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big) + \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i\big(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0)\big)\right](t-t_0)^2, i = 1,2, \dots, n. \tag{12}$$

This procedure can be repeated till the arbitrary order coefficients of RPS solutions are obtained. Moreover, higher accuracy can be achieved by evaluating more components of the solution.

Theorem 4.1. Suppose that $x_i(t), i = 1,2, \dots, n$ are the exact solutions for system of FIVPs (1) and (2). Then, the approximate solutions obtained by the RPS technique are just the Taylor expansion of $x_i(t), i = 1,2, \dots, n$.

Corollary 4.1. [4] If some of $x_i(t), i = 1,2, \dots, n$ is a polynomial, then the RPS technique will be obtained the exact solution.

It will be convenient to have a notation for the error in the approximation $x_i(t) \approx x_i^k(t)$. Accordingly, we will let $\text{Rem}_i^k(t)$ denote the difference between $x_i(t)$ and its kth Taylor polynomial; that is,

$$\text{Rem}_i^k(t) = x_i(t) - x_i^k(t) = \sum_{m=k+1}^{\infty} \frac{x_i^{(m)}(t_0)}{m!}(t-t_0)^m, i = 1,2,\dots,n,$$

where the functions $\text{Rem}_i^k(t)$ are called the kth-remainder for the Taylor series of $x_i(t)$.

5. Numerical Experiments

In this section, we show by example that the system of crisp IVPs can be modeled in a natural way as system of FIVPs. To illustrate this, consider the dynamic supply and demand system. The system of ODE corresponding to this problem is $p'(t) = \theta - k_1(s - s_0)$ and $s'(t) = k_2(p - p_0)$ where p is the price, s is the supply, p_0 is the equilibrium price, s_0 is equilibrium supply, θ is the rate of inflation, and k_1, k_2 are positive constant corresponding to the dynamic nature of the system. Here, we are considering an item such that increasing its price p results in an increase in supply s but that increasing its supply s will ultimately decrease its price p. Furthermore, we will assume there are two factors that influence price; inflation and supply. The factor $s - s_0$ means that; firstly, if $s > s_0$, the supply is too large and price is to decrease; secondly, if $s < s_0$, supply is too low and price tends to increase, while on the other hand, the factor $p - p_0$ means that; firstly, if $p > p_0$, price is high and supply increasing; Secondly, if $p < p_0$, price is low and supply decreases. Uncertainty in determining the initial values, inaccuracy in element modeling, and other parameters cause uncertainty in the system. Considering them instead as system of FIVPs yields more realistic results.

Example [40] Consider the following dynamic supply and demand differential system of fuzzy equations on [0,1]:

$$\begin{aligned} p'(t) &= \theta - k_1(s - s_0), \\ s'(t) &= k_2(p - p_0), \end{aligned} \tag{13}$$

subject to the fuzzy initial conditions

$$x_1(0) = \alpha_1, x_2(0) = \alpha_2, \tag{14}$$

where $\alpha_{1r} = [20 + 5r, 30 - 5r]$ and $\alpha_{2r} = [550 + 50r, 650 - 50r]$.

For numerical results and comparisons, the following values, for parameters, are considered: $\theta = 0.05, s_0 = 1200, p_0 = 25,$ and $k_1 = k_2 = 0.5$. The exact fuzzy solutions of system of FIVP (13) and (14) in parametric form are

$$\begin{aligned} p_r(t) &= \left[\left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}},\left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{45}{2}r\right)e^{\frac{t}{2}}\right] \\ &\quad +\frac{6001}{10}\sin\left(\frac{x}{2}\right)+25, \\ s_r(t) &= \left[\left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}},\left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}}\right] \\ &\quad -\frac{6001}{10}\cos\left(\frac{x}{2}\right)+\frac{12001}{10}. \end{aligned} \tag{15}$$

To apply the RPS method, put $p_r(t)=[p_{1r}(t),p_{2r}(t)]$ and $s_r(t)=[s_{1r}(t),s_{2r}(t)]$. Then we have the following system of ODE:

$$\begin{aligned} p'_{1r}(t) &= 0.05-0.5(s_{2r}-1200), \\ p'_{2r}(t) &= 0.05-0.5(s_{1r}-1200), \\ s'_{1r}(t) &= 0.5(p_{1r}-25), \\ s'_{2r}(t) &= 0.5(p_{2r}-25), \end{aligned} \tag{16}$$

subject to the initial conditions

$$\begin{aligned} &p_{1r}(0)=20+5r, p_{2r}(0)=30-5r, \\ &s_{1r}(0)=550+50r, s_{2r}(0)=650-50r. \end{aligned} \tag{17}$$

Using RPS method, taking $c_{1,0}=20+5r$, $c_{2,0}=30-5r$, $c_{3,0}=550+50r$, and $c_{4,0}=650-50r$ as initial guess approximation, the Taylor series expansions of solutions for Eqs. (16) and (17) are as follows

$$\begin{aligned} p_{1r}(t) &= 20+5r+\sum_{m=1}^{\infty}c_{1,m}t^m, \\ p_{2r}(t) &= 30-5r+\sum_{m=1}^{\infty}c_{2,m}t^m, \\ s_{1r}(t) &= 550+50r+\sum_{m=1}^{\infty}c_{3,m}t^m, \\ s_{2r}(t) &= 650-50r+\sum_{m=1}^{\infty}c_{4,m}t^m, \end{aligned} \tag{18}$$

in which the kth residual functions are given as

$$\begin{aligned} \mathrm{Res}_{1,r}^k(t) &= \sum_{m=1}^{k}mc_{1,m}t^{m-1}-\left[0.05-0.5\left(\left(650-50r+\sum_{m=1}^{k}c_{4,m}t^m\right)-1200\right)\right], \\ \mathrm{Res}_{2,r}^k(t) &= \sum_{m=1}^{k}mc_{1,m}t^{m-1}-\left[0.05-0.5\left(\left(550+50r+\sum_{m=1}^{k}c_{3,m}t^m\right)-1200\right)\right], \\ \mathrm{Res}_{3,r}^k(t) &= \sum_{m=1}^{k}mc_{3,m}t^{m-1}-\left[0.5\left(\left(20+5r+\sum_{m=1}^{k}c_{1,m}t^m\right)-25\right)\right], \\ \mathrm{Res}_{4,r}^k(t) &= \sum_{m=1}^{k}mc_{4,m}t^{m-1}-\left[0.5\left(\left(30-5r+\sum_{m=1}^{k}c_{2,m}t^m,\right)-25\right)\right], \end{aligned} \tag{19}$$

Anyhow, when $N = 10$ is used throughout the computations; the following are the first fifth terms of RPS approximation of Eqs. (16) and (17):

$$\begin{aligned}
p_{1r}(t) &= \left(\frac{45}{2}-\frac{45}{2}r\right)\left[1-\frac{t}{2}+\frac{t^2}{2^2 2!}-\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}-\frac{t^5}{2^5 5!}\right] \\
&\quad -\left(\frac{55}{2}-\frac{55}{2}r\right)\left[1+\frac{t}{2}+\frac{t^2}{2^2 2!}+\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}+\frac{t^5}{2^5 5!}\right] \\
&\quad +\frac{6001}{10}\left[\frac{t}{2}-\frac{t^3}{2^3 3!}+\frac{t^5}{2^5 5!}\right]+25, \\
p_{2r}(t) &= \left(\frac{45}{2}r-\frac{45}{2}\right)\left[1-\frac{t}{2}+\frac{t^2}{2^2 2!}-\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}-\frac{t^5}{2^5 5!}\right] \\
&\quad +\left(\frac{55}{2}-\frac{45}{2}r\right)\left[1+\frac{t}{2}+\frac{t^2}{2^2 2!}+\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}+\frac{t^5}{2^5 5!}\right] \\
&\quad +\frac{6001}{10}\left[\frac{t}{2}-\frac{t^3}{2^3 3!}+\frac{t^5}{2^5 5!}\right]+25, \\
s_r(t) &= \left(\frac{45}{2}r-\frac{45}{2}\right)\left[1-\frac{t}{2}+\frac{t^2}{2^2 2!}-\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}-\frac{t^5}{2^5 5!}\right] \\
&\quad -\left(\frac{55}{2}-\frac{55}{2}r\right)\left[1+\frac{t}{2}+\frac{t^2}{2^2 2!}+\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}+\frac{t^5}{2^5 5!}\right] \\
&\quad -\frac{6001}{10}\left[1-\frac{t^2}{2^2 2!}+\frac{t^4}{2^4 4!}\right]+\frac{12001}{10}, \\
s_r(t) &= \left(\frac{45}{2}-\frac{45}{2}r\right)\left[1-\frac{t}{2}+\frac{t^2}{2^2 2!}-\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}-\frac{t^5}{2^5 5!}\right] \\
&\quad +\left(\frac{55}{2}-\frac{55}{2}r\right)\left[1+\frac{t}{2}+\frac{t^2}{2^2 2!}+\frac{t^3}{2^3 3!}+\frac{t^4}{2^4 4!}+\frac{t^5}{2^5 5!}\right] \\
&\quad -\frac{6001}{10}\left[1-\frac{t^2}{2^2 2!}+\frac{t^4}{2^4 4!}\right]+\frac{12001}{10}.
\end{aligned} \tag{18}$$

If we collect the above results, then the exact solutions of Eqs. (16) and (17) have the general form which are coinciding with the exact solutions

$$\begin{aligned}
p_{1r}(t) &= \left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}}+\frac{6001}{10}\sin\left(\frac{x}{2}\right)+25, \\
p_{2r}(t) &= \left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{45}{2}r\right)e^{\frac{t}{2}}+\frac{6001}{10}\sin\left(\frac{x}{2}\right)+25, \\
s_r(t) &= \left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}}-\frac{6001}{10}\cos\left(\frac{x}{2}\right)+\frac{12001}{10}, \\
s_r(t) &= \left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}}-\frac{6001}{10}\cos\left(\frac{x}{2}\right)+\frac{12001}{10}.
\end{aligned} \tag{19}$$

As result, the fuzzy exact solution of FIVP (13) and (14) in parametric form can be written equivalently in the form of the following:

$$p_r(t) = \left[\left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}},\left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{45}{2}r\right)e^{\frac{t}{2}}\right] + \frac{6001}{10}\sin\left(\frac{x}{2}\right)+25,$$

$$s_r(t) = \left[\left(\frac{45}{2}r-\frac{45}{2}\right)e^{-\frac{t}{2}}-\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}},\left(\frac{45}{2}-\frac{45}{2}r\right)e^{-\frac{t}{2}}+\left(\frac{55}{2}-\frac{55}{2}r\right)e^{\frac{t}{2}}\right] - \frac{6001}{10}\cos\left(\frac{x}{2}\right)+\frac{12001}{10}. \qquad (20)$$

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