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Semi-Boolean Corner Rings

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Abstract

We show that if R is a ring with an arbitrary idempotent e such that both eRe and $(1-e)R(1-e)$ are semi-boolean rings, then $R/J(R)$ is a nil-clean ring. In particular, under certain additional circumstances, R is also nil-clean. These results somewhat improve on achievements due to Diesl in J. Algebra (2013), Koşan-Wang-Zhou in J. Pure Appl. Algebra (2016) and Danchev in Bull. Iran. Math. Soc. (2017).

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1. Introduction and Background

Throughout the present article all rings R under consideration shall be assumed to be associative with identity element 1, which is different from the zero element 0. Standardly, $Id(R)$ stands for the set of all idempotents of R and $Nil(R)$ for the set of all nilpotents of R. As usual, $U(R)$ denotes the group of all units in R and $J(R)$ denotes the Jacobson radical of R. Note that $1+J(R) \subseteq U(R)$ is always fulfilled. We also use E_{ij} to denote the $n \times n$ matrix with (i,j)-entry 1 and the other entries 0. All unexplained traditional notions and notations may be found in [12]. For instance, a ring R is called *boolean*, provided $R = Id(R)$.

About the specific terminology, recall that the *prime radical* $P(R)$ of a ring R is defined as the intersection of all prime ideals in R (note that it coincides with the lower nil-radical $Nil_*(R)$). A ring R is called *2-primal* if $P(R) = Nil(R)$. Notice that every commutative ring as well as every reduced ring (i.e., a ring with no non-zero nilpotents) has to be 2-primal. We recall also that a ring R has a *bounded index of nilpotence* if there is $n \in \mathbb{N}$ such that $a^n = 0$ for every $a \in Nil(R)$. Moreover, the upper nil-radical $Nil^*(R)$ of R is defined as the sum of all two-sided nil-ideals in R and thus it is the largest nil-ideal of R. In conclusion, it follows that the inclusions $Nil_*(R) = P(R) \subseteq Nil^*(R) \subseteq Nil(R) \cap J(R)$ are true.

On the other vein, we shall say that a ring R is *J-primal* if $P(R) = J(R)$. Obvious examples of J-primal rings are those rings in which each prime ideal is maximal – e.g., commutative regular rings.

Moreover, a ring R is known to be *J-reduced* if $Nil(R) \subseteq J(R)$. It thus follows that all 2-primal rings are J-reduced. About the converse, all J-primal J-reduced rings are obviously 2-primal. Likewise, if R is 2-primal and $J(R)$ is nil, then R is J-primal.

The following two fundamental concepts were defined in [14].

Definition 1.1. A ring R is called *exchange* if, for each $x \in R$, there exists $e \in Id(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$.

Definition 1.2. A ring R is called *clean* if, for each $x \in R$, there exist $u \in U(R)$ and $e \in Id(R)$ such that $x = u + e$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is said to be *strongly clean*.

It is clear that abelian (in particular, commutative) clean rings are always strongly clean.

In [9] was introduced the following major concept.

Definition 1.3. A ring R is called *nil-clean* if, for each $r \in R$, there are $q \in Nil(R)$ and $e \in Id(R)$ with $r = q + e$. If, in addition, the commutativity condition $qe = eq$ is satisfied, the nil-clean ring R is said to be *strongly nil-clean*.

It is obvious that abelian (in particular, commutative) nil-clean rings are always strongly nil-clean. But it was independently established in [7] and [11] by exploiting different ideas that a ring R is strongly nil-clean if, and only if, $J(R)$ is nil and $R/J(R)$ is boolean.

The next important notion was stated in [15].

Definition 1.4. A ring R is called *semi-boolean* if, for each $r \in R$, there are $j \in J(R)$ and $e \in Id(R)$ with $r = j + e$. If, in addition, the commutativity condition $je = ej$ is satisfied, the semi-boolean ring is said to be *strongly semi-boolean*.

It was proved in [15] that a ring R is semi-boolean if, and only if, $R/J(R)$ is boolean and all idempotents in R lift modulo $J(R)$.

So, the following containment holds:

boolean $\Rightarrow$ *strongly nil-clean* $\Rightarrow$ *strongly clean* $\Rightarrow$ *semi-boolean* $\Rightarrow$ *clean* $\Rightarrow$ *exchange*.

Two significant lines of research in noncommutative ring theory are to find to what extent the ring-theoretic properties of R are preserved by its corner ring eRe, where $e \in Id(R)$, or by its full $n \times n$ matrix ring $\mathbb{M}_n(R)$, where $n \in \mathbb{N}$, and visa versa. The most important principal known results in these two subjects are these: It was established in [14] that, for any idempotent e of R, the ring R is exchange if, and only if, both eRe and $(1-e)R(1-e)$ are exchange rings. Also, it was proved in [10] that if eRe and $(1-e)R(1-e)$ are both clean rings, then R is a clean ring. However, it was constructed in [16] a clean ring R of characteristic 2 for which eRe is not a clean ring. Nevertheless, it was obtained in [3] that if R is a strongly clean ring, then eRe is again a strongly clean ring. Moreover, it was shown in [9, Corollary 3.26] that if R is a strongly nil-clean ring, then eRe is a strongly nil-clean ring. Likewise, this was extended in [7] to the so-called UU rings which are rings whose units are the sum of 1 and a nilpotent. So, a rather natural question which immediately arises is what we can say about the ring structure of R, provided that both eRe and $(1-e)R(1-e)$ are semi-boolean rings. We will somewhat settle this in the sequel, thus extending some of the results in [6].

On the other hand, it was proved in [14] and [10] that if R is an exchange ring, respectively a clean ring, then the same is $\mathbb{M}_n(R)$. Besides, it was obtained in [1, Corollary 7] that if R is a commutative nil-clean ring, then the ring $\mathbb{M}_n(R)$ is nil-clean. This was extended in [11, Theorem 6.1] to 2-primal strongly nil-clean rings and in [11, Corollary 6.8] to strongly nil-clean rings of bounded index of nilpotence. Some more substantial generalizations were established in [6], too.

The objective of this paper is to continue the study on these two directions. Our further work is organized in the next two sections as follows:

2. Main Results

The following technical claim is our non-trivial key instrument (cf. [6], as well).

Lemma 2.1. *Suppose that R is a ring with $e \in Id(R)$ for which both eRe and $(1-e)R(1-e)$ are boolean rings. Then R is nil-clean.*

Proof. For any $r \in R$, the equality $r = ere+(1-e)r(1-e)+(1-e)re+er(1-e)$ holds. Note that $ere \in eRe$ and $(1-e)r(1-e) \in (1-e)R(1-e)$ are both orthogonal idempotents, while $(1-e)re$ and $er(1-e)$ are nilpotents of order 2, because $[(1-e)re]^2 = (1-e)re.(1-e)re = 0 = er(1-e).er(1-e) = [er(1-e)]^2$. On the other side, putting $t = (1-e)re + er(1-e)$ and $f = (1-e)rer(1-e) + er(1-e)re$, one sees that $t^2 = f$. But by assumption both $(1-e)rer(1-e) \in (1-e)R(1-e)$ and $er(1-e)re \in eRe$ are idempotents, so that f is again an idempotent being the sum of two orthogonal idempotents. Therefore, $t^2 = f^2$, i.e., $t^2 - f^2 = 0$. Besides, one checks that $tf = (1-e)rer(1-e)re + er(1-e)rer(1-e) = ft$ and so $(t-f)(t+f) = 0$. Since $2f = 0$ as f is an element of the sum of two boolean rings, the last equality is equivalent to $(t-f)^2 = 0$, that is, $t \in f + Nil(R)$. Next, observing that $r = ere + (1-e)r(1-e) + t$, one may also write that

$$r = [ere + er(1-e)re] + [(1-e)r(1-e) + (1-e)rer(1-e)] + q,$$

where $q \in Nil(R)$. Since $e_1 = ere + er(1-e)re = e(r + r(1-e)r)e \in eRe$ and $e_2 = (1-e)r(1-e) + (1-e)rer(1-e) = (1-e)(r+rer)(1-e) \in (1-e)R(1-e)$ are idempotents with zero products $e_1.e_2 = e_2.e_1 = 0$, one can infer that $e_1 + e_2 = e'$ is again an idempotent. Consequently, since we may represent r like $r = e' + q$ with $e' \in Id(R)$ and $q \in Nil(R)$, we finally obtain by definition that R is nil-clean, as asserted. □

Remark 1. It is worthwhile noticing that it cannot be expected that such a ring R will be strongly nil-clean. In fact, it was demonstrated in [9] that every unit in a strongly nil-clean ring must be a unipotent, that is, the sum of 1 and a nilpotent element. However, in the matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ over the boolean ring $\mathbb{Z}_2$, the matrix unit $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ cannot be a unipotent because the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is never a nilpotent. Even more, the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is a unit having the inverse $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Now we can derive our first basic assertion.

Theorem 2.2. *Suppose that R is a ring with $e \in Id(R)$ for which both eRe and $(1-e)R(1-e)$ are semi-boolean rings. Then $R/J(R)$ is a nil-clean ring.*

Proof. Owing to the mentioned above result from [15], accomplishing it with [12], for any $h \in Id(R)$ we deduce that the factor-ring $hRh/J(hRh) = hRh/hJ(R)h \cong h'(R/J(R))h'$ is boolean for some idempotent $h' = h + J(R)$ of $R/J(R)$. Hence Lemma 2.1 successfully applies to get that $R/J(R)$ is nil-clean, as stated. □

The next formula is our basic omnibus (cf. [4] and [6], too).

Lemma 2.3. *For each ring R and each idempotent e, the following equality is fulfilled:*

$$P(eRe) = eP(R)e.$$

Proof. First, observe that if P is any prime ideal of R then either $ePe = eRe$, or ePe is a prime ideal of eRe. Therefore, $eP(R)e$ is an intersection of some of the prime ideals of eRe, so it is a semiprime ideal of eRe. This means that $P(eRe) \subseteq eP(R)e$.

To show that the converse inclusion is valid, it is enough to prove that $eP(R)e \subseteq Q$ for every prime ideal Q of eRe. We shall obtain this by demonstrating that $Q = ePe$ for some prime ideal P of R. To establish that, note that the set $X = eRe \setminus Q$ is what McCoy called in [13] an "m-system" of eRe:

it is nonempty, and for any $x, y \in X$, there is some $a \in eRe$ such that $xay \in X$. Notice however that X is also an m-system even in R, and that X is disjoint from the ideal RQR. Let $P \supseteq RQR$ be an ideal of R which is maximal with respect to being disjoint from X. In [13] was proved that any such ideal has to be prime. Since P is disjoint from X, we must have $P \cap eRe = Q$, and consequently $ePe = Q$, as desired. □

We are now in a position to illustrate the truthfulness of the following:

Corollary 2.4. *If R is a J-primal ring, then for any idempotent e of R the corner eRe is a J-primal ring, as well.*

Proof. Utilizing [12] together with Lemma 2.3, one has that $J(eRe) = eJ(R)e = eP(R)e = P(eRe)$, as needed. □

Now, we have all the ingredients to prove the following statement.

Theorem 2.5. *Suppose that R is a ring with $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both J-primal semi-boolean rings. Then R is nil-clean.*

Proof. We foremost see that for f being either e or $1-e$, the formula $J(fRf) = P(fRf)$ is true. Furthermore, one observes with the aid of Lemma 2.3 along with [7] that the ring

$$fRf/J(fRf) = fRf/P(fRf) = fRf/fP(R)f \cong f'(R/P(R))f'$$

for $f' = f + P(R) \in Id(R/P(R))$, is boolean, so that Lemma 2.1 leads us to the fact that $R/P(R)$ is nil-clean. Since $P(R)$ is a nil-ideal of R, we consequently consulting with [9] conclude that R is nil-clean, as pursued. □

The next consequence follows by an immediate combination of Corollary 2.4 and Theorem 2.5.

Corollary 2.6. *Suppose R is a J-primal ring whose corners eRe and $(1-e)R(1-e)$ are semi-boolean rings. Then R is a nil-clean ring.*

Remark 2. First of all, note the important fact that J-primal semi-boolean rings are exactly the 2-primal strongly nil-clean rings, and reciprocally. In fact, since J-primal semi-boolean rings have nil Jacobson ideals and these rings modulo their Jacobson radicals are boolean, one of the aforementioned chief results from [7] is applicable to deduce that they are strongly nil-clean. But then the isomorphism $U(R)/[1+J(R)] \cong U(R/J(R)) = 1$ applies to infer that $U(R) = 1+J(R)$ and so $1+Nil(R) \subseteq U(R)$ gives that $Nil(R) = J(R) = P(R)$ (see [2, Theorem 2.3(2)] too), whence we obtain the wanted 2-primariness. Conversely, as discussed above, strongly nil-clean rings are themselves semi-boolean. Also, in view of [9], Jacobson radicals have to be nil and thereby the 2-primariness now routinely implies J-primariness, as required.

In this aspect, it is worth noticing that the essence in the proofs of the last theorem and its corresponding analogue from [6] is that in both versions of

J-primal rings and 2-primal rings it must be that $P(fRf) = Nil^*(fRf) = J(fRf)$ for the idempotent $f \in R$ defined as above.

On the other side, appealing to Theorem 2.2, we deduce that the factor-ring $R/J(R)$ is nil-clean. In order to prove that R is also nil-clean, with [9] at hand, it suffices to show directly that $J(R)$ is nil. To that goal, it could be applied the classical Pierce's direct decomposition for $J(R)$ into subrings

$$J(R) = eJ(R)e \oplus (1-e)J(R)e \oplus eJ(R)(1-e) \oplus (1-e)J(R)(1-e)$$

used above or, in accordance with [12], we may also use the Pierce's matrix representation

$$J(R) \cong \begin{pmatrix} eJ(R)e & eJ(R)(1-e) \\ (1-e)J(R)e & (1-e)J(R)(1-e) \end{pmatrix} = \begin{pmatrix} J(eRe) & eJ(R)(1-e) \\ (1-e)J(R)e & J((1-e)R(1-e)) \end{pmatrix},$$

which tells us about the corresponding matrix structure. However, without the extra assumption that R is commutative, we perhaps will need the validity of the famous Köthe's conjecture.

Utilizing ordinary induction arguments in the pivotal Lemma 2.1, all statements concerning corners eRe and $(1-e)R(1-e)$ can be expanded to statements on a system of mutually orthogonal idempotents $\{e_i\}_{i=1}^n$ with $1 = e_1 + \cdots + e_n$ such that all corners e_iRe_i are as above in the case of two idempotents (compare with [10] and [6], too).

With this at hand, we now arrive at the next assertion (compare with [1] and [6], as well).

Corollary 2.7. *Let R be a J-primal semi-boolean ring. Then $\mathbb{M}_n(R)$ is nil-clean for each $n \geq 1$.*

Proof. Knowing that $R \cong E_{11}\mathbb{M}_n(R)E_{11} \cong \cdots \cong E_{nn}\mathbb{M}_n(R)E_{nn}$ for any $n \geq 1$, where $\{E_{ii}\}_{i=1}^n$ forms a complete system of matrix idempotents (i.e., a set of matrix orthogonal idempotents with sum 1), it suffices to employ the generalized form of Theorem 2.5 to get the desired claim. □

Remark 3. We cannot expect that, for each $n \geq 2$, $\mathbb{M}_n(R)$ is semi-boolean for the semi-boolean ring R, because it can be showed as in [15] that $\mathbb{M}_n(R)$ is semi-boolean if, and only if, $R = J(R)$. However, this is nonsense when $1 \in J(R)$ since it would imply that $0 = 1 - 1 \in U(R)$, and hence $0 = 1$, that is impossible. For a more general situation with given details, the interested readers can see [5].

3. Left-Open Problems

We close with two problems of interest. We shall say that a ring R is *2-nil-clean* if each its element is the sum of two idempotents and a nilpotent.

Problem 1. If R is a ring for which both eRe and $(1-e)R(1-e)$ are commutative semi-boolean rings, does it follow that R is 2-nil-clean? In particular, if R is commutative semi-boolean, is then $\mathbb{M}_n(R)$ 2-nil-clean?

A ring is said to be *weakly boolean* if any its element is an idempotent or minus an idempotent (see also [5]). Generalizing this, in [8] were introduced *weakly nil-clean* rings as those rings whose elements are the sum or the difference of a nilpotent and an idempotent. We are now ready to state the following:

Problem 2. If R is a ring for which both eRe and $(1-e)R(1-e)$ are weakly boolean rings, is it true that R is weakly nil-clean?

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