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Heronian Mean Labeling of Graphs

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Abstract

A function f is called a **Heronian Mean Labeling** of a graph **G = (V,E)** with p vertices and q edges if it is possible to label the vertices $\mathbf{x \in V}$ with distinct labels $\mathbf{f(x)}$ from **1,2,..,q+1** in such a way that when each edge $\mathbf{e = uv}$ is labeled with

$$\mathbf{f(e = uv)} = \left\lceil \frac{\mathbf{f(u)} + \sqrt{\mathbf{f(u)f(v)}} + \mathbf{f(v)}}{\mathbf{3}} \right\rceil (\mathbf{OR}) \left\lfloor \frac{\mathbf{f(u)} + \sqrt{\mathbf{f(u)f(v)}} + \mathbf{f(v)}}{\mathbf{3}} \right\rfloor$$

then the edge labels are distinct. In this case, *f* is a Heronian mean labeling of G and G is called a **Heronian Mean Graph.** In this paper, we prove that Path, Cycle, Comb, Dragon, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$ ($n \leq 5$), Complete Graph K_n ($n \leq 4$) are Heronian Mean Graphs.

Mathematics Subject Classification: 05C78

Keywords: Graph, Heronian Mean Graph, Path, Cycle, Comb, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$, Complete Graph K_n

1. Introduction

By a graph we mean a finite undirected graph without loops or parallel edges. For all detailed survey of graph labeling, we refer to J.A. Gallian [1]. For all other standard terminology and notations we follow Harary [2]. The concept of Mean labeling has been introduced by S. Somasundaram and R. Ponraj [3] in 2004. S.Somasundaram and S.S. Sandhya introduced Harmonic mean labeling [4] in 2012. Motivated by the above works we introduce a new type of labeling called **Heronian Mean Labeling.**

In this paper we investigate the Heronian Mean Labeling of Path, Cycle, Comb, Dragon, Ladder, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$, Complete Graph K_n.We will provide a brief summary of definitions and other information which are necessary for our present investigation.

A Path P_n is a walk in which all the vertices are distinct. A **Cycle C_n** is a Closed Path. The graph obtained by joining a single pendant edge to each vertex of a Path is called a **Comb**. A **Triangular Snake T_n** is obtained from a path $u_1,u_2,\dots\dots u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n-1$. That is every edge of a path is replaced by a triangle C_3. A **Quadrilateral Snake Q_n** is obtained from a path $u_1,u_2,\dots\dots u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i respectively and then joining v_i and w_i. That is every edge of a path is replaced by a cycle C_4. The **square G^2** of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever $d(u, v) \leq 2$ in G. **A Complete Bipartite** graph $K_{m,n}$ is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 is joined to all the vertices of V_2, Where $|V_1| = m$ and $|V_2| = n$. **A Star** graph is the complete bipartite graph $K_{1,n}$. A graph **G** is said to be **Complete**, if every pair of its distinct vertices are adjacent. A Complete Graph on **n** vertices is denoted by K_n.

Definition 1.1:

A graph **G=(V,E)** with p vertices and q edges is said to be a **Heronian Mean graph** if it is possible to label the vertices **x∈V** with distinct labels $f(x)$ from **1,2,…,q+1** in such a way that when each edge $e = uv$ is labeled with,

$$f(e = uv) = \left\lceil \frac{f(u) + \sqrt{f(u)f(v)} + f(v)}{3} \right\rceil (OR) \left\lfloor \frac{f(u) + \sqrt{f(u)f(v)} + f(v)}{3} \right\rfloor$$

then the edge labels are distinct. In this case **f** is called a **Heronian Mean labeling** of G.

Remark: 1.2

If G is a Heronian mean graph, then ‘1’ must be a label of one of the vertices of G, Since an edge should get label ‘1’.

Remark: 1.3

If u gets label '1', then any edge incident with u must get label 1 (or) 2 (or) 3. Hence this vertex must have a degree ≤ 3.

2. Main Results

Theorem: 2.1

Any Path $\mathbf{P_n}$ is a Heronian mean graph.

Proof:

Let P_n be a path $u_1u_2u_3\ldots\ldots u_n$.

Define a function $f: V(P_n) \to \{1,2,3,\ldots\ldots,q+1\}$ by $f(u_i) = i, 1 \leq i \leq n$.
Then the edge labels are $f(u_iu_{i+1}) = i, 1 \leq i \leq n-1$
Hence $\mathbf{P_n}$ is a Heronian mean graph.

Example 2.2: A Heronian mean labeling of $\mathbf{P_8}$ is shown below.

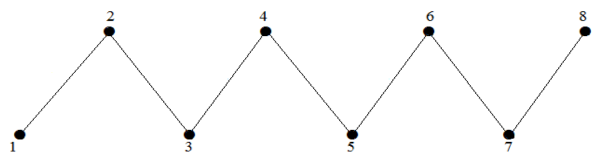


Figure: 1

Theorem: 2.3

For $\boldsymbol{n \geq 3}$, Any Cycle $\mathbf{C_n}$ is a Heronian mean graph.

Proof:

Let $\mathbf{C_n}$ be a cycle of length n. Let the cycle be $u_1u_2u_3\ldots\ldots u_nu_1$

$$\text{Take } n = \begin{cases} 2m, \text{if n is even.} \\ 2m+1, \text{if n is odd.} \end{cases}$$

Define a function $f: V(\mathbf{C_n}) \to \{1,2,3,\ldots,q+1\}$ by $f(u_i) = 2i-1,\ 1 \leq i \leq m+1$.

$$f(u_{m+j}) = \begin{cases} n-2j+3, 2 \leq j \leq m, \text{if n is even.} \\ n-2j+4, 2 \leq j \leq m+1, \text{if n is odd.} \end{cases}$$

The set of labels of edges of $\mathbf{C_n}$ are $\{1,2,3,\ldots,n\}$.
Obviously,f is a Heronian mean labeling. Hence $\mathbf{C_n}$ is a Heronian mean graph.

Example 2.4: A Heronian mean labeling of $\mathbf{C_6}$ is shown below.

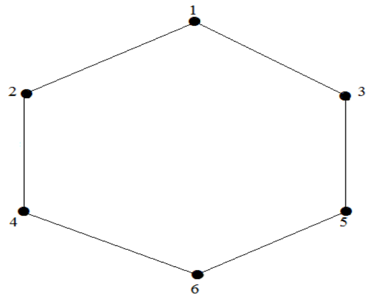


Figure: 2

Theorem: 2.5

Any Comb $\mathbf{P_n}$ $\mathbf{\Theta}$ $\mathbf{K_1}$ is a Heronian mean graph.

Proof:

Let $P_n \ \Theta\ K_1$ be a comb obtained from a path $P_n = u_1u_2 \ldots . u_n$ by joining a vertex u_i to $v_i(1\leq i \leq n)$.
Define a function, $f: V(P_n\ \Theta\ K_1) \to \{1,2,3, \ldots \ldots . q+1\}$ by $f(u_i) = 2i\ , 1 \leq i \leq n$

$$f(v_i) = 2i - 1\ , \quad 1 \leq i \leq n$$

Edges are labeled by, $f(u_iu_{i+1}) = 2i\ ,\ 1 \leq i \leq n-1$

$$f(u_iv_i) = 2i - 1\ , 1 \leq i \leq n$$

Clearly, f is a Heronian mean labeling. Hence $P_n\ \Theta\ K_1$ is a Heronian mean graph.

Example 2.6: A Heronian mean labeling of $\boldsymbol{P_5}\ \boldsymbol{\Theta K_1}$ is shown below.

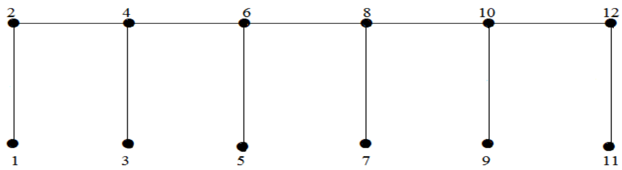


Figure: 3

Theorem: 2.7

Any Triangular Snake $\boldsymbol{T_n}$ is a Heronian mean graph.

Proof:

Let T_n be a Triangular Snake. Let u_i, v_i be the vertices of a Triangular Snake. Join u_iv_i and $u_{i+1}v_i, 1 \leq i \leq n-1$.
Define a function, $f: V(\ T_n\ ) \to \{1,2,3, \ldots \ldots, q+1\}$by $f(u_i) = 3i - 2\ , 1 \leq i \leq n$,

$$f(v_i) = 3i - 1\ , 1 \leq \mathrm{i} \leq n - 1\ .$$

Edges are labeled by, $f(u_i u_{i+1}) = 3i - 1, 1 \leq i \leq n - 1$,

$$f(u_i v_i) = 3i - 2 , 1 \leq i \leq n - 1 ,$$

$$f(u_{i+1} v_i) = 3i , 1 \leq i \leq n - 1 .$$

Clearly, f is a Heronian mean labeling. Hence $\boldsymbol{T_n}$ is a Heronian mean graph.

Example 2.8: A Heronian mean labeling of $\boldsymbol{T_5}$ is shown below.

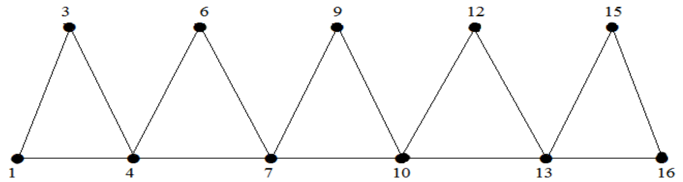


Figure: 4

Theorem: 2.9

Any Quadrilateral Snake $\boldsymbol{Q_n}$ is a Heronian mean graph.

Proof:

Let Q_n be a quadrilateral snake.

Define a function, $f: V(Q_n) \rightarrow \{1,2,3, \ldots\ldots, q+1\}$ by $f(u_i) = 4i - 3 , 1 \leq i \leq n$,

$$f(v_i) = 4i - 2 , 1 \leq i \leq n - 1,$$

$$f(w_i) = 4i , 1 \leq i \leq n - 1.$$

Edges are labeled by, $f(u_i u_{i+1}) = 4i - 2 , \forall\ 1 \leq i \leq n - 1$,

$$\mathrm{f(u_i v_i)} = 4\mathrm{i} - 3 , \forall\ 1 \leq \mathrm{i} \leq \mathrm{n} - 1 ,$$

$$\mathrm{f(u_{i+1} w_i)} = 4\mathrm{i} , \forall\ 1 \leq \mathrm{i} \leq \mathrm{n} - 1,$$

$$\mathrm{f(v_i w_i)} = 4\mathrm{i} - 1 , \forall\ 1 \leq \mathrm{i} \leq \mathrm{n} - 1.$$

Clearly f is a Heronian mean labeling. Hence $\mathbf{Q_n}$ is a Heronian mean graph.

Example 2.10: A Heronian mean labeling of $\mathbf{Q_5}$ is shown below.

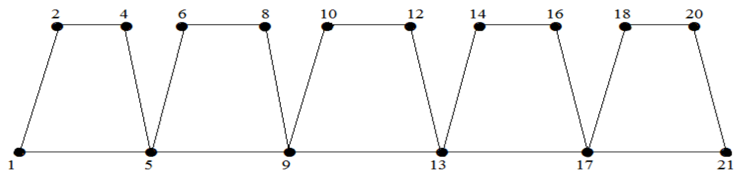


Figure: 5

Theorem: 2.11

Star $\mathbf{K_{1,n}}$ is a Heronian mean graph if and only if $\mathbf{n \leq 5}$.

Proof:

Star $\mathbf{K_{1,1}}$ is same as $\mathbf{P_2}$ and Star $\mathbf{K_{1,2}}$ is same as $\mathbf{P_3}$. Clearly $\mathbf{K_{1,1}}$ and $\mathbf{K_{1,2}}$ are Heronian Mean graphs. Let the central vertex of the Star be **u**, and the other vertices be $v_1v_2v_3 \ldots\ldots v_n$. respectively.

Case (i):

When $\mathbf{2 \leq i \leq 5}$, assign the label 3 to the vertex u and **i** ($\mathbf{1 \leq i \leq 2}$) to the vertex $\mathbf{v_i}$. Then label **3+i** to $\mathbf{v_{2+i}}$, $\mathbf{1 \leq i \leq 3}$. Clearly the above labeling pattern is a Heronian Mean Labeling, Which is shown in the **figure: 6.**

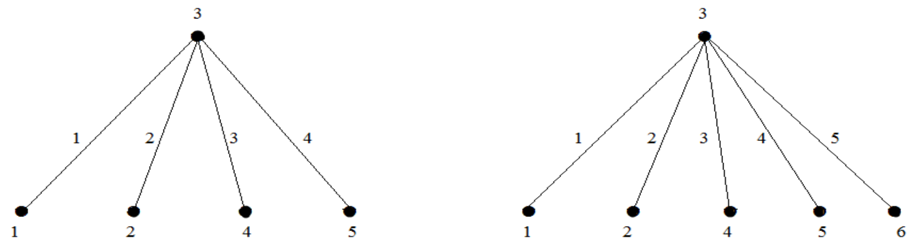


Figure: 6

Case (ii):

Assume $\mathbf{n} > 5$ and suppose $\mathbf{K_{1,n}}$ has Heronian Mean labeling.

Here we consider the following subcases.

Subcase (ii)(a):

Let the label of the central vertex be **u** be 2.

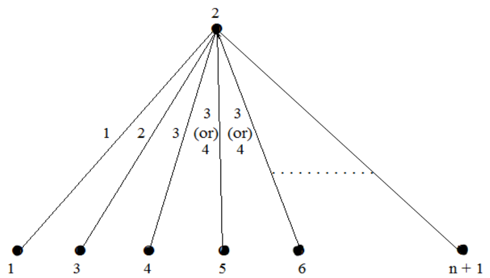


Figure: 7

The other vertices $v_1,v_3,v_4,v_5,v_6,v_7\ldots$. are labeled as 1,3,4,5,6,7,… respectively. Here the edge labels of $\mathbf{uv_4}$ is 3 itself and the edge labels of $\mathbf{uv_5}$ and $\mathbf{uv_6}$ are from 3 and 4. This is not possible.

Subcase (ii)(b):

Let the label of the central vertex be **u** be 4.

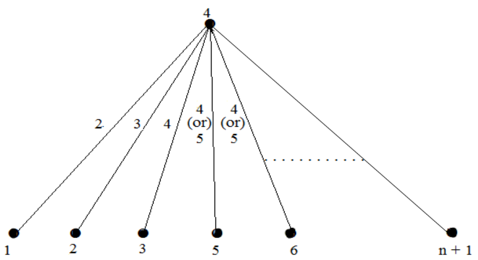


Figure:8

The other vertices $v_1,v_2,v_3,v_5,v_6,v_7\ldots$ are labeled as 1,2,3,5,6,7,… respectively. In this case there will be no edge with label 1. Here the edge labels of $\mathbf{uv_3}$ is 4 and the edge label of $\mathbf{uv_5}$ and $\mathbf{uv_6}$ is 4 and 5. This is not possible.
From all these, we conclude that $K_{1,n}$, $n > 5$ is not a Heronian Mean Graph.
Now we investigate the Heronian mean labeling of complete graphs.

Theorem: 2.12

The graph $\boldsymbol{P_n^{(2)}}$ is a Heronian mean graph.

Proof:

Let P_n be the path $u_1, u_2, \ldots \ldots u_n$. Clearly $P_n^{(2)}$ has n vertices and $2n-3$ edges.
Define a function, $f: V\left(P_n^{(2)}\right) \to \{1,2,\ldots\ldots q+1\}$ by $f(u_i) = 2i-1$, $1 \le i \le n-1$.

$$f(u_n) = 2n-2.$$

Edges are labeled by, $f(u_i u_{i+1}) = 2i-1$, $1 \le i \le n-1$

$$f(u_i u_{i+1}) = 2i \ , 1 \le i \le n-2$$

$$f(u_{n-1}u_n = 2n-3 \ ,$$

Hence f is a Heronian mean labeling of $\boldsymbol{P_n^{(2)}}$.

Example 2.13: A Heronian mean labeling of $P_7^{(2)}$ is given below.

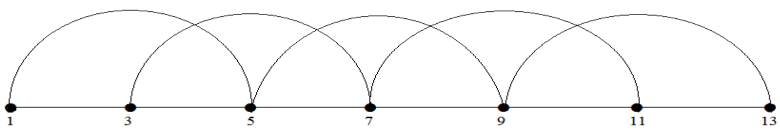


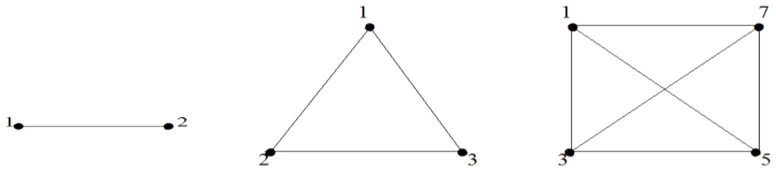
Figure: 9

Theorem: 2.14

If $\mathbf{n} > 4$, $\mathbf{K_n}$ is not a Heronian mean graph.

Proof:

Clearly $\mathbf{K_1}$ is a Heronian mean graph**.** By Theorem 2.1 and Theorem 2.2, Clearly $\mathbf{K_2}$ **and** $\mathbf{K_3}$ are Heronian Mean graphs. Also $\mathbf{K_4}$ is also a Heronian mean graph. The labeling pattern of $\mathbf{K_2}, \mathbf{K_3}$ **and** $\mathbf{K_4}$ are given below.



$\mathbf{K_2}$ $\mathbf{K_3}$ $\mathbf{K_4}$

If n > 4**,** We have repetition of edge labels, which is not possible. Hence to get the edge label 1, we need a vertex u with label 1. There are four more vertices u_1, u_2, u_3, u_4 incident with u. This is not possible by Remark 1.2. Hence $\mathbf{K_n}$ **, n** > 4 is not a Heronian Mean graph.

Remark: 2.15

If G is a k-regular graph (with k>4), then G is not Heronian.

3. Conclusion

The Study of labeled graph is important due to its diversified applications. All graphs are not Heronian mean graphs. It is very interesting to investigate graphs which admit Heronian Mean Labeling. In this paper, we proved that Path, Cycle, Comb, Triangular Snake, Quadrilateral Snake, Star, Square of a Path and Complete Graph are Heronian Mean Graphs. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. It is possible to investigate similar results for several other graphs.

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