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The Solitary Wave Solution of the Swift-Hohenberg Equation Using He's Semi Inverse Method

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Abstract

In this paper we solved the Swift-Hohenberg equation in one spatial dimension using the He's semi-inverse method. We found eighteen families of solutions.

Keywords: Swift-Hohenberg equation, He's semi-inverse method

1 Introduction

The Swift-Hohenberg equation (SHEq), has been very useful to study pattern formation from Localized hexagonpatterns [1], till Snaking of radial solutions in multidimensions [2]. In general, pattern formation is in the core of phase transitions and non-equilibrium Physics [3], and SHEq is one of the main equations that has proven its worthiness in this research field, [4]-[5]. This equation is:

$$\frac{\partial U}{\partial t} + (1 + \frac{\partial^2}{\partial x^2})^2 U - \mu U + U^3 = 0 \qquad (1)$$

On the other hand, He's semi-inverse method has become a very interesting variational technique in order to find solitary wave solutions [6]. Basically, we start from a trial functional that we make stationary in order to find the solutions. In our case, the trial functional for SHEq is:

$$J(u) = \int \left(\frac{1}{2}(u^{''})^2 - (u^{'})^2 + (\frac{1-\alpha}{2})u^2 - \frac{1}{4}u^4 \right) d\xi \tag{2}$$

2 Solution 1

We select:

$$u = \sqrt{a_1} sech(b\xi) \tag{3}$$

Therefore, replacing each one of the integrands, we obtain:

$$-\left(\frac{\partial u}{\partial \xi}\right)^2 = -a_1 b^2 sech^2(b\xi)\tanh^2(b\xi) \tag{4}$$

$$\frac{1}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 = \frac{b^2}{8} a_1 (\cosh(2b\xi) - 3)^2 sech^6(b\xi) \tag{5}$$

$$(\frac{1-\alpha}{2})u^2 - \frac{1}{4}u^4 = (\frac{1-\alpha}{2})a_1 sech^2(b\xi) - \frac{1}{4}a_1^2 sech^4(b\xi) \tag{6}$$

Now, we calculate the integrals:

$$J_1 = -a_1 b^2 \int_0^\infty sech^2(b\xi)\tanh^2(b\xi) d\xi = -a_1 b^2 \frac{1}{3b} \tag{7}$$

$$J_2 = \int_0^\infty \frac{b^2}{8} a_1 (\cosh(2b\xi) - 3)^2 sech^6(b\xi) d\xi = \frac{a_1 b^2}{8}\frac{28}{15b} \tag{8}$$

$$J_3 = \int_0^\infty ((\frac{1-\alpha}{2})a_1 sech^2(b\xi) - \frac{1}{4}a_1^2 sech^4(b\xi)) d\xi = (\frac{1-\alpha}{2b})a_1 - \frac{2}{12b}a_1^2 \tag{9}$$

Then, the entire action is:

$$J(a_1, b) = J_1 + J_2 + J_3 = -\frac{a_1 b}{3} + \frac{7a_1 b}{30} + (\frac{1-\alpha}{2b})a_1 - \frac{a_1^2}{6b} \tag{10}$$

We make stationary J using a_1 and b, we have:

$$\frac{\partial J}{\partial a_1} = 0; \qquad \frac{\partial J}{\partial b} = 0 \tag{11}$$

We get two nonlinear algebraic equations for a_1 and b:

$$-\frac{b^2}{10} + (\frac{1-\alpha}{2}) - \frac{a_1}{3} = 0, \quad -\frac{b^2}{10} - (\frac{1-\alpha}{2}) + \frac{a_1}{6} = 0 \tag{12}$$

Then, doing some algebra, the solutions for the field u corresponds to:

$$u_{1,2} = \sqrt{2(1-\alpha)}sech(\pm i\sqrt{\frac{5(1-\alpha)}{3}}\xi) \tag{13}$$

3 Solution 2

We select:

$$u = a_1 sech^2(b\xi) \tag{14}$$

Therefore, replacing in eq. (2), we obtain:

$$-\left(\frac{\partial u}{\partial \xi}\right)^2 = -a_1^2 4b^2 tanh^2(b\xi)sech^4(b\xi) \tag{15}$$

$$\frac{1}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 = a_1^2 2b^4(cosh(2bx) - 2)^2 sech^8(bx) \tag{16}$$

$$(\frac{1-\alpha}{2})u^2 - \frac{1}{4}u^4 = (\frac{1-\alpha}{2})a_1^2 sech^4(b\xi) - \frac{1}{4}a_1^4 sech^8(b\xi) \tag{17}$$

Now, we calculate the integrals:

$$J_1 = -4a_1 b^2 \int_0^\infty tanh^2(b\xi)sech^4(b\xi)d\xi = -4a_1 b^2 \frac{1}{b}\frac{\pi^2}{90} \tag{18}$$

$$J_2 = a_1^2 2b^4 \int_0^\infty (cosh(2bx) - 2)^2 sech^8(bx)d\xi = a_1^2 \frac{16b^4}{21b} \tag{19}$$

$$J_3 = \int_0^\infty ((\frac{1-\alpha}{2})a_1^2 sech^4(b\xi) - \frac{1}{4}a_1^4 sech^8(b\xi))d\xi = (\frac{1-\alpha}{3b})a_1^2 - \frac{4}{35b}a_1^4 \quad (20)$$

Then, the entire action is:

$$J(a_1, b) = J_1 + J_2 + J_3 = -4a_1 b\frac{\pi^2}{90} + a_1^2\frac{16b^3}{21} + (\frac{1-\alpha}{3b})a_1^2 - \frac{4}{35b}a_1^4 \quad (21)$$

We make stationary J using a_1 and b, obtaining two nonlinear algebraic equations. Then, we get:

$$\frac{\partial J}{\partial a_1} = -4b^2\frac{\pi^2}{90} + a_1\frac{32b^4}{21} + 2(\frac{1-\alpha}{3})a_1 - \frac{4}{35}a_1^3 = 0 \quad (22)$$

$$\frac{\partial J}{\partial b} = -4\frac{\pi^2 b^2}{90} + a_1\frac{16b^4}{7} - (\frac{1-\alpha}{3})a_1 + \frac{4}{35}a_1^3 = 0 \quad (23)$$

Defining

$$g_1 = (\frac{8}{21})^2, \quad g_2 = (\frac{56}{21})(1-\alpha), \quad g_3 = 4(1-\alpha) \quad (24)$$
$$g_4 = (\frac{8\pi^2}{90})^2(\frac{8}{25}), \quad g_5 = (\frac{8\pi^2}{90})^2(\frac{21}{48})(1-\alpha)$$

$$l_1 = -2g_2^3 + 9g_1g_2g_3 + 9g_1g_2g_4 - 27g_1^2g_5 + (4\left(-g_2^2 + 3g_1(g_3+g_4)\right)^3 \quad (25)$$
$$+ \left(-2g_2^3 + 9g_1g_2g_3 + 9g_1g_2g_4 - 27g_1^2g_5\right)^2)^{1/2}$$

Doing some algebra, we get:

$$a_{1,1} = -(-\frac{g_2}{3g_1} + \frac{2^{1/3}g_2^2}{3g_1(l_1)^{1/3}} - \frac{2^{1/3}g_3}{(l_1)^{1/3}} - \frac{2^{1/3}g_4}{(l_1)^{1/3}} + \frac{(l_1)^{1/3}}{32^{1/3}g_1})^{1/2} \quad (26)$$

$$a_{1,2} = (-\frac{g_2}{3g_1} + \frac{2^{1/3}g_2^2}{3g_1(l_1)^{1/3}} - \frac{2^{1/3}g_3}{(l_1)^{1/3}} - \frac{2^{1/3}g_4}{(l_1)^{1/3}} + \frac{(l_1)^{1/3}}{32^{1/3}g_1})^{1/2} \quad (27)$$

$$a_{1,3} = -(-\frac{g_2}{3g_1} - \frac{g_2^2}{3\ 2^{2/3}g_1(l_1)^{1/3}} + \frac{ig_2^2}{2^{2/3}\sqrt{3}g_1(l_1)^{1/3}} + \frac{g_3}{2^{2/3}(l_1)^{1/3}} \quad (28)$$
$$-\frac{i\sqrt{3}g_3}{2^{2/3}(l_1)^{1/3}} + \frac{g_4}{2^{2/3}(l_1)^{1/3}} - \frac{i\sqrt{3}g_4}{2^{2/3}(l_1)^{1/3}} - \frac{(l_1)^{1/3}}{62^{1/3}g_1} - \frac{i(l_1)^{1/3}}{22^{1/3}\sqrt{3}g_1})^{1/2}$$

$$a_{1,4} = (-\frac{g_2}{3g_1} - \frac{g_2^2}{32^{2/3}g_1\left(l_1\right)^{1/3}} + \frac{ig_2^2}{2^{2/3}\sqrt{3}g_1\left(l_1\right)^{1/3}} + \frac{g_3}{2^{2/3}\left(l_1\right)^{1/3}} \tag{29}$$
$$-\frac{i\sqrt{3}g_3}{2^{2/3}\left(l_1\right)^{1/3}} + \frac{g_4}{2^{2/3}\left(l_1\right)^{1/3}} - \frac{i\sqrt{3}g_4}{2^{2/3}\left(l_1\right)^{1/3}} - \frac{\left(l_1\right)^{1/3}}{62^{1/3}g_1} - \frac{i\left(l_1\right)^{1/3}}{22^{1/3}\sqrt{3}g_1})^{1/2}$$

$$a_{1,5} = -(-\frac{g_2}{3g_1} - \frac{g_2^2}{32^{2/3}g_1\left(l_1\right)^{1/3}} - \frac{ig_2^2}{2^{2/3}\sqrt{3}g_1\left(l_1\right)^{1/3}} + \frac{g_3}{2^{2/3}\left(l_1\right)^{1/3}} \tag{30}$$
$$+\frac{i\sqrt{3}g_3}{2^{2/3}\left(l_1\right)^{1/3}} + \frac{g_4}{2^{2/3}\left(l_1\right)^{1/3}} + \frac{i\sqrt{3}g_4}{2^{2/3}\left(l_1\right)^{1/3}} - \frac{\left(l_1\right)^{1/3}}{62^{1/3}g_1} + \frac{i\left(l_1\right)^{1/3}}{22^{1/3}\sqrt{3}g_1})^{1/2}$$

$$a_{1,6} = (-\frac{g_2}{3g_1} - \frac{g_2^2}{32^{2/3}g_1\left(l_1\right)^{1/3}} - \frac{ig_2^2}{2^{2/3}\sqrt{3}g_1\left(l_1\right)^{1/3}} + \frac{g_3}{2^{2/3}\left(l_1\right)^{1/3}} \tag{31}$$
$$+\frac{i\sqrt{3}g_3}{2^{2/3}\left(l_1\right)^{1/3}} + \frac{g_4}{2^{2/3}\left(l_1\right)^{1/3}} + \frac{i\sqrt{3}g_4}{2^{2/3}\left(l_1\right)^{1/3}} - \frac{\left(l_1\right)^{1/3}}{62^{1/3}g_1} + \frac{i\left(l_1\right)^{1/3}}{22^{1/3}\sqrt{3}g_1})^{1/2}$$

and

$$b_{1,2} = \pm\sqrt{\pm\sqrt{(\frac{21}{48})(1-\alpha) - \frac{8}{35}a_1^2}} \tag{32}$$

Then, we get 12 families of solutions.

4 Solution 3

Finally, we choose:

$$u = a_1 e^{(-b\xi^2)} \tag{33}$$

Therefore, replacing in each one of the integrands in eq. (2), we get:

$$-\left(\frac{\partial u}{\partial \xi}\right)^2 = -4a_1^2 b^2 \xi^2 e^{(-2b\xi^2)} \tag{34}$$

$$\frac{1}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 = 2a_1^2 b^2 e^{(-2b\xi^2)}(2b\xi^2 - 1)^2 \tag{35}$$

$$(\frac{1-\alpha}{2})u^2 - \frac{1}{4}u^4 = (\frac{1-\alpha}{2})a_1^2 e^{(-2b\xi^2)} - \frac{1}{4}a_1^4 e^{(-4b\xi^2)} \tag{36}$$

Now, we calculate the integrals:

$$J_1 = -4a_1^2 b^2 \int_0^\infty \xi^2 e^{(-2b\xi^2)} d\xi = -4a_1^2 b^2 \sqrt{\frac{\pi}{2}} \frac{1}{8b^{3/2}} \tag{37}$$

$$J_2 = 2a_1^2 b^2 \int_0^\infty e^{(-2b\xi^2)} (2b\xi^2 - 1)^2 d\xi = \frac{3}{4} a_1^2 b^{3/2} \sqrt{\frac{\pi}{2}} \tag{38}$$

$$\begin{aligned} J_3 = & \int_0^\infty ((\frac{1-\alpha}{2})a_1^2 e^{(-2b\xi^2)} - \frac{1}{4}a_1^4 e^{(-4b\xi^2)}) d\xi \\ & = \sqrt{\frac{\pi}{2}} \frac{(1-\alpha)a_1^2}{4\sqrt{b}} - \frac{\sqrt{\pi}}{16} \frac{a_1^4}{\sqrt{b}} \end{aligned} \tag{39}$$

Then, the entire action is:

$$\begin{aligned} J(a_1, b) = J_1 + J_2 + J_3 = -4a_1^2 b^2 \sqrt{\frac{\pi}{2}} \frac{1}{8b^{3/2}} + \frac{3}{4} a_1^2 b^{3/2} \sqrt{\frac{\pi}{2}} \\ + \sqrt{\frac{\pi}{2}} \frac{(1-\alpha)a_1^2}{4\sqrt{b}} - \frac{\sqrt{\pi}}{16} \frac{a_1^4}{\sqrt{b}} \end{aligned} \tag{40}$$

Making stationary J, we get two nonlinear algebraic equations for a_1 and b. Then:

$$\frac{\partial J}{\partial a_1} = -b + \frac{3}{2}b^2 + \frac{(1-\alpha)}{2} - \frac{\sqrt{2}}{4}a_1^2 = 0 \tag{41}$$

$$\frac{\partial J}{\partial b} = -b + \frac{9}{2}b^2 - \frac{(1-\alpha)}{2} + \frac{\sqrt{2}}{8}a_1^2 = 0 \tag{42}$$

$$l_2 = 169 - 26\alpha + \alpha^2, \quad l_3 = (5+\alpha)^2 \left(16 - 14\alpha + \alpha^2\right) \tag{43}$$

$$a_{1,7} = -\sqrt{\frac{2}{3}}(-\frac{21\sqrt{2}}{l_2} - \frac{93\sqrt{2}\alpha}{l_2} + \frac{45\sqrt{2}\alpha^2}{l_2} - \frac{3\sqrt{2}\alpha^3}{l_2} - \frac{4\sqrt{6}\sqrt{l_3}}{l_2})^{1/2} \tag{44}$$

$$b_7 = (6 - 12\alpha + 6\alpha^2 + \frac{546}{l_2} + \frac{2376\alpha}{l_2} - \frac{1356\alpha^2}{l_2} + \frac{168\alpha^3}{l_2} - \frac{6\alpha^4}{l_2} + \frac{104\sqrt{3}\sqrt{l_3}}{l_2} - \frac{8\sqrt{3}\alpha\sqrt{l_3}}{l_2})/(60 + 12\alpha) \tag{45}$$

$$a_{1,8} = \sqrt{\frac{2}{3}}(-\frac{21\sqrt{2}}{l_2} - \frac{93\sqrt{2}\alpha}{l_2} + \frac{45\sqrt{2}\alpha^2}{l_2} - \frac{3\sqrt{2}\alpha^3}{l_2} - \frac{4\sqrt{6}\sqrt{(l_3}}{l_2})^{1/2} \tag{46}$$

$$b_8 = (6 - 12\alpha + 6\alpha^2 + \frac{546}{l_2} + \frac{2376\alpha}{l_2} - \frac{1356\alpha^2}{l_2} + \frac{168\alpha^3}{l_2} - \frac{6\alpha^4}{l_2} + \frac{104\sqrt{3}\sqrt{l_3}}{l_2} - \frac{8\sqrt{3}\alpha\sqrt{l_3}}{l_2})/(60 + 12\alpha) \tag{47}$$

$$a_{1,9} = -\sqrt{\frac{2}{3}}(-\frac{21\sqrt{2}}{l_2} - \frac{93\sqrt{2}\alpha}{l_2} + \frac{45\sqrt{2}\alpha^2}{l_2} - \frac{3\sqrt{2}\alpha^3}{l_2} + \frac{4\sqrt{6}\sqrt{l_3}}{l_2})^{1/2} \tag{48}$$

$$b_9 = (6 - 12\alpha + 6\alpha^2 + \frac{546}{l_2} + \frac{2376\alpha}{l_2} - \frac{1356\alpha^2}{l_2} + \frac{168\alpha^3}{l_2} - \frac{6\alpha^4}{l_2} - \frac{104\sqrt{3}\sqrt{l_3}}{l_2} + \frac{8\sqrt{3}\alpha\sqrt{l_3}}{l_2})/(60 + 12\alpha) \tag{49}$$

$$a_{1,10} = \sqrt{\frac{2}{3}}(-\frac{21\sqrt{2}}{l_2} - \frac{93\sqrt{2}\alpha}{l_2} + \frac{45\sqrt{2}\alpha^2}{l_2} - \frac{3\sqrt{2}\alpha^3}{l_2} + \frac{4\sqrt{6}\sqrt{l_3}}{l_2})^{1/2} \tag{50}$$

$$b_{10} = (6 - 12\alpha + 6\alpha^2 + \frac{546}{l_2} + \frac{2376\alpha}{l_2} - \frac{1356\alpha^2}{l_2} + \frac{168\alpha^3}{l_2} - \frac{6\alpha^4}{l_2} - \frac{104\sqrt{3}\sqrt{l_3}}{l_2} + \frac{8\sqrt{3}\alpha\sqrt{l_3}}{l_2})/(60 + 12\alpha) \tag{51}$$

Then, we get 4 families of solutions.

5 Conclusions

This paper presents the He's semi-inverse method applied to the Swift-Hohenberg equation in one dimension. We find 18 families of solutions. They are:

$$u_{1,2} = \sqrt{2(1-\alpha)}sech(\pm\sqrt{\frac{5(1-\alpha)}{3}}\xi) \tag{52}$$

$$u_i = a_{1,i}sech^2(b_i\xi) \tag{53}$$

$$u_j = a_{1,j}e^{(-b_j\xi^2)} \tag{54}$$

Where, $i = 1, ..., 12$ and $j = 1, ..., 4$. Finally, the extension to 2 or 3 dimensions is straightforward.

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