International Mathematical Forum, Vol. 12, 2017, no. 16, 747 - 753
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.7652

Permanents and Determinants of Tridiagonal Matrices with (s,t)-Pell Numbers

Hasan Huseyin Gulec

Eregli Faculty of Education
Necmettin Erbakan University, Konya, Turkey

Abstract

In this study, we define a $n \times n$ tridiagonal matrix which have elements of (s,t)-Pell numbers and then investigate the determinantal properties.

Keywords: determinant, permanent, tridiagonal matrix.

1 Introduction

The Fibonacci, Lucas, Pell and Pell-Lucas sequences have been discussed in so many articles and books (see [2-5]). For $n > 1$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$ sequences are defined as $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$, $P_n = 2P_{n-1} + P_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $P_0 = 0$, $P_1 = 1$ and $Q_0 = 2$, $Q_1 = 2$. Further details about the Pell and Pell-Lucas numbers can be seen in [1].

In [6], Kılıç gave the definition of generalized Pell (p,i)-numbers and then presented their generating matrix. He obtained relationships between the generalized Pell (p,i)-numbers and their sums and permanents of certain matrices. Also, he derived the generalized Binet formulas, sums, combinatorial representations. In [7,8], the authors defined a new matrix generalization of the Fibonacci and Lucas numbers, and using essentially a matrix approach they showed properties of these matrix sequences. In [9], the authors gave the following definition.

For any real numbers (s,t) and $n \geqslant 2$, let $s^2+t>0$, $s>0$ and $t \neq 0$. The (s,t)-Pell sequence $\{p_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)-Pell Lucas sequence $\{q_n(s,t)\}_{n\in\mathbb{N}}$ are defined respectively by

$$p_n(s,t) = 2sp_{n-1}(s,t) + tp_{n-2}(s,t), \tag{1}$$

$$q_n(s,t) = 2sq_{n-1}(s,t) + tq_{n-2}(s,t) \tag{2}$$

with initial conditions $p_0(s,t)=0$, $p_1(s,t)=1$ and $q_0(s,t)=2$, $q_1(s,t)=2s$. Then considering these sequences, they defined the matrix sequences which have elements of (s,t)-Pell and (s,t)-Pell Lucas sequences and investigated their properties.

The permanent of an n-square matrix is defined by

$$perA = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n [15].

Let $A=[a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, ..., r_m$. We say A contractible on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. Let we consider the following result (see [16]): Let A be a nonnegative integral matrix of order $n>1$ and let B be a contraction of A. Then

$$perA = perB. \tag{3}$$

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [10] defined a $n \times n$ super diagonal $(0,1)-$matrix $F(n,k)$ for $n > k \geq 2$ and show that the permanent of $F(n,k)$ equals to the generalized order-k Fibonacci numbers. Also he gave some relations involving the permanents of some $(0,1)-$Circulant matrices and the usual Fibonacci numbers.

In [11], the authors presented a nice result involving the permanent of an $(-1,0,1)-$matrix and the Fibonacci number F_{n+1}. The authors then explored similar directions involving the positive subscripted Fibonacci and Lucas Numbers as well as their uncommon negatively subscripted counterparts. Finally the authors explore the generalized order-k Lucas numbers, (see [12] and [13] for more detail the generalized Fibonacci and Lucas numbers), and their permanents.

In [14], Lehmer proved a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries were somewhat arbitrary.

2 Main Results

In this section, we define a tridiagonal matrix and then show that the permanent and determinant of this matrix equal to the (s,t)-Pell number.

Definition 1 *We define a $n \times n$ tridiagonal $(1,2s,t)-$matrix $G_n(s,t) = [g_{ij}]$ with $g_{ii} = 2s,\ g_{i+1,i} = 1,\ g_{i,i+1} = t$ for $1 \le i \le n$ and 0 otherwise. That is,*

$$G_n(s,t) = \begin{bmatrix} 2s & t & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix}. \tag{4}$$

Then we give following Theorem.

Theorem 2 *Let the matrix $G_n(s,t)$ be as in (4). Then for $n \ge 1$,*

$$perG_n(s,t) = perG_n^{n-2}(s,t) = p_{n+1}(s,t)$$

where p_n is the nth (s,t)-Pell number.

Proof. If $n = 1$, then $perG_1 = per[2s] = 2s = p_2$.

If $n = 2$, then

$$G_2 = \begin{bmatrix} 2s & t \\ 1 & 2s \end{bmatrix}$$

and hence $perG_2 = 4s^2 + t = p_3$.

Let G_n^r be rth contraction of G_n, $1 \le r \le n-2$. From the definition of G_n, the matrix G_n can be contracted on column 1 so that

$$G_n^1 = \begin{bmatrix} 4s^2+t & 2st & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix}.$$

Since the matrix G_n^1 can be contracted on column 1 and $p_3 = 4s^2+t$, $tp_2 = 2st$.

$$G_n^2 = \begin{bmatrix} 8s^3+4st & 4s^2t+t^2 & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix} = \begin{bmatrix} p_4 & tp_3 & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix}.$$

Furthermore, the matrix G_n^2 can be contracted on column 1 so that

$$G_n^3 = \begin{bmatrix} p_5 & tp_4 & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix}.$$

Continuing this process, we obtain

$$G_n^r = \begin{bmatrix} p_{r+2} & tp_{r+1} & & & 0 \\ 1 & 2s & t & & \\ & 1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & 1 & 2s \end{bmatrix}$$

for $3 \le r \le n-4$. Hence,

$$G_n^{n-3} = \begin{bmatrix} p_{n-1} & tp_{n-2} & 0 \\ 1 & 2s & t \\ 0 & 1 & 2s \end{bmatrix}$$

which, by contraction of G_n^{n-3} on column 1, gives

$$G_n^{n-2} = \begin{bmatrix} 2sp_{n-1}+tp_{n-2} & tp_{n-1} \\ 1 & 2s \end{bmatrix} = \begin{bmatrix} p_n & tp_{n-1} \\ 1 & 2s \end{bmatrix}.$$

By the Eq. (3) and the definition of the (s,t)-Pell numbers, we obtain

$$perG_n = perG_n^{n-2} = 2sp_n + tp_{n-1} = p_{n+1}.$$

So the proof is complete. ■

Lemma 3 [11] *Let $C_1(n)$ be $n \times n$ tridiagonal matrix. That is,*

$$C_1(n) = \begin{bmatrix} c_{1,1} & c_{1,2} & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & \\ & c_{3,2} & c_{3,3} & \ddots & \\ & & \ddots & \ddots & c_{n-1,n} \\ & & & c_{n,n-1} & c_{n,n} \end{bmatrix}$$

where the sign of the main diagonal of the matrix $C_1(n)$ is positive. Then the successive permanents of $C_1(n)$ are given by the recursive formula

$$\begin{aligned} perC_1(1) &= c_{1,1}, \\ perC_1(2) &= c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\ perC_1(n) &= c_{n,n}perC_1(n-1) + c_{n-1,n}c_{n,n-1}perC_1(n-2). \end{aligned}$$

Now, we can give a different way to prove Theorem 2 by taking advantage of Lemma 3.

$$\begin{aligned} perG_1 &= 2s = p_2, \\ perG_2 &= 2s2s + t = 2sp_2 + tp_1 = p_3, \\ perG_3 &= 2sp_3 + tp_2 = p_4, \\ &\vdots \\ perG_n &= 2sp_n + tp_{n-1} = p_{n+1}. \end{aligned}$$

Hence the result.

[17] A matrix A is called convertible if there is an $n \times n$ $(1,-1)-$matrix K such that $perA = det(A \circ K)$, where $A \circ K$ denotes the Hadamard product of A and K. Such a matrix K is called a converter of A.

Let S be a $(1,-1)-$matrix of order n, defined by

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}.$$

Now we denote the matrix $G_n \circ S$. Thus

$$G_n \circ S = \begin{bmatrix} 2s & t & & & 0 \\ -1 & 2s & t & & \\ & -1 & 2s & \ddots & \\ & & \ddots & \ddots & t \\ 0 & & & -1 & 2s \end{bmatrix}.$$

Then we have

$$\det(G_n \circ S) = perG_n = p_{n+1}.$$

Acknowledgement

A small summary of the study was presented at the *4. International Conference on Matrix Analysis and Applications* (ICMAA2013) held in Konya in 2013.

References

[1] M. Bicknell, A Primer on The Pell Sequence and Related Sequences, *Fibonacci Quarterly,* **13** (1975), 345-349.

[2] A. T. Benjamin, S. S. Plott and J. A. Sellers, Tiling proofs of recent sum identities involving Pell numbers, *Annals of Combinatorics,* **12** (2008), 271-278. https://doi.org/10.1007/s00026-008-0350-5

[3] N. Taskara, K. Uslu, H. H. Gulec, On the properties of Lucas numbers with binomial coefficients, *Applied Mathematics Letters,* **23** (2010), no. 1, 68-72. https://doi.org/10.1016/j.aml.2009.08.007

[4] Hasan Huseyin Gulec, *Pell Matrix Sequences and Their Properties*, Ph.D. Thesis, Selcuk University, Konya, 2014.

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons Inc, NY, 2001. https://doi.org/10.1002/9781118033067

[6] E. Kılıç, The generalized Pell (p, i)-numbers and their Binet formulas, combinatorial representations, sums, *Chaos, Solitions & Fractals,* **40** (2009), 2047-2063. https://doi.org/10.1016/j.chaos.2007.09.081

[7] H. Civciv and R. Türkmen, On the (s, t)-Fibonacci and Fibonacci matrix sequences, *Ars Combinatoria,* **87** (2008), 161-173.

[8] H. Civciv and R. Türkmen, Notes on the (s, t)-Lucas and Lucas matrix sequences, *Ars Combinatoria,* **89** (2008), 271-285.

[9] Hasan Huseyin Gulec, Necati Taskara, On the (s, t)-Pell and (s, t)-Pell–Lucas sequences and their matrix representations, *Applied Mathematics Letters,* **25** (2012), 1554-1559. https://doi.org/10.1016/j.aml.2012.01.014

[10] H. Minc, Permanents of $(0, 1)$-Circulants, *Canad. Math. Bull.*, **7** (1964), 253-263. https://doi.org/10.4153/cmb-1964-023-3

[11] E. Kilic and D. Tasci, On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, *Rocky Mountain J. Math.*, **37** (2007), no. 6, 1953-1969. https://doi.org/10.1216/rmjm/1199649832

[12] E. Kilic and D. Tasci, On the generalized order-k Fibonacci and Lucas numbers, *Rocky Mountain J. Math.*, **36** (2006), no. 6, 1915-1926. https://doi.org/10.1216/rmjm/1181069352

[13] D. Tasci and E. Kilic, On the order-k generalized Lucas numbers, *Appl. Math. Comput.*, **155** (2004), no. 3, 637-641.
https://doi.org/10.1016/s0096-3003(03)00804-x

[14] D. Lehmer, Fibonacci and related sequences in periodic tridiagonal matrices, *Fib. Quart.*, **13** (1975), 150-158.

[15] H. Minc, *Permanents*, Encyclopedia Math. Appl., Vol. 6, Addison–Wesley Publishing Company, London, 1978.

[16] R. A. Brualdi & P. M. Gibson, Convex polyhedra of Doubly Stochastic matrices I: Applications of the Permanent Function, *J. Combin. Theory Series A*, **22** (1977), 194-230.
https://doi.org/10.1016/0097-3165(77)90051-6

[17] E. Kılıç, D. Taşçı, Negatively subscripted Fibonacci and Lucas numbers and their complex factorizations, *Ars Combin.*, **96** (2010), 275-288.

Received: July 7, 2017; Published: August 5, 2017