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Numerical Solutions for Handling Systems of Obstacle Boundary Value Problems Using the RPS Method

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Abstract

In this paper, we apply the residual power series technique to find out the solutions for systems of obstacle boundary value problems of second order. This technique is effective and easy to use for solving such systems without linearization, perturbation, or discretization. The solutions are provided in the form of a rapidly convergent series with easily computable components using symbolic computation software. The proposed method obtains the expansion of the solutions of the systems under appropriate initial guesses approximations in the form of polynomials. The numerical comparisons are presented and discussed quantitatively to illustrate the solutions. Numerical results show the potentiality, the generality, and the superiority of our algorithm for solving such systems.

1. Introduction

The variational inequality theory has been developed since the early sixties, which has greatly stimulated the research in pure and applied sciences. The variational inequalities arise in models for a large number of mathematical, physical, regional, engineering, and other problems [1-3]. The theory of the variational inequalities has led to exciting and important contributions to pure and applied sciences which includes work on differential equations, contact problems in elasticity, fluid flow through porous media, general equilibrium problems in economics and transportation, unilateral obstacle, moving and free boundary problems.

In the variational formulation of such problems, the location of the contact area becomes an intrinsic part of the solution and no special techniques are needed to obtain it. If the obstacle is known, then the variational inequalities can be characterized by a system of differential equations by using the penalty function technique [3]. In general, it is not possible to obtain the analytical solution of systems of differential equations, obtained from obstacle, unilateral, moving and free boundary value problems (BVPs), and problems of the defection of plates. In fact, many of real physical phenomena encountered, are almost impossible to solve, these problems must be attacked by various approximate and numerical methods.

The main advantage of the RPS method is the simplicity in computing the coefficients of terms of the series solutions by using the differential operators only and not as the other well-known analytic techniques that need the integration operators which is difficult in general [4-9]. Moreover, the proposed method can be easily applied in the spaces of higher dimension solutions and can be applied without any limitation on the nature of the systems and the type of classifications. Numerical techniques are widely used by scientists and engineers to solve their problems. A major advantage for numerical techniques is that a numerical answer can be obtained even when a problem has no analytical solution. On the other hand, many applications for different problems by using other numerical algorithms can be found in [10-18].

In this paper, we using the residual power series method (RPSM) to develop a new numerical method for obtaining smooth approximations to solutions and their derivatives for systems of obstacle problems of second order. This paper is organized as follows. In Section 2, a short description for the RPS is presented. In Section 3, we discuss the problem of the study. In Section 4, we present some numerical results. Finally, conclusions are given in Section 5.

2. The Residual Power Series Method

Series expansions are very important aids in numerical calculations, especially for quick estimates made in hand calculation, for example, in evaluating functions, integrals, or derivatives. Since, the advent of computers, it has, however, become more common to treat the differential and integral problems directly, using different approximation method instead of series expansions. But in connection with the development of automatic methods for formula manipulation, one can anticipate renewed interest for series methods. These methods have some advantages, especially in multidimensional solution [19-23]. In the present paper, we invested the residual error concept in the power series technique to obtain a simple procedure to find out the coefficients of the series solutions for systems of obstacle BVPs of different orders. The RPSM is effective and easy to use for solving systems of obstacle BVPs of different classifications. Different from the classical power series approach, the RPS technique does not need to compare the coefficients of the corresponding terms and recursion relations are not required. This technique computes the coefficient of the power series by a chain of linear equations. The RPS technique is different from the traditional higher order Taylor series approach.

The Taylor series approach is computationally expensive for large orders. By using residual error concept, we get series solutions, in practice truncated series solutions.

This paper will add new analytical method, so-called, the RPSM, to approximate the solutions for systems of obstacle BVPs of different orders.

- Firstly, we extend the application of the RPSM to construct solution for system of second order obstacle BVPs of the following form [24-28]:

$$y''(x) = \begin{cases} f(x), & a \le x \le c, \\ y(x)g(x) + f(x) + r, & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{2.1}$$

subject to the boundary conditions

$$y(a) = \alpha \;\&\; y(b) = \beta. \tag{2.2}$$

3. Construct the Solutions for Second Order Obstacle Problem

In this section, the RPSM is used to seek the solution for system of second order obstacle problem. Here, we construct the solution by substituting their residual power series expansion among their truncated residual functions. From the resulting equations; recursion formulas for the computation of the coefficients are derived, while the coefficients in the expansions can be computed recursively by recurrent differentiating of the truncated residual functions by means of the symbolic computation software used. On the other hand, applications for different problems using numerical methods can be found in [29-38]. To apply the RPSM, we need to find out the first few approximations terms for the following problem:

$$y''(x) = \begin{cases} f(x), & a \le x \le c, \\ y(x)g(x) + f(x) + r, & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{3.1}$$

subject to the boundary conditions

$$y(a) = \alpha_1 \;\&\; y(b) = \alpha_2. \tag{3.2}$$

Here, we assuming the continuity conditions of y and y' at c and d. Further, f and g are continuous functions on $[a,b]$ and $[c,d]$, respectively. The parameters r, α_1, and α_2 are real finite constants. Anyhow, we have three cases in order to obtain the approximate solution depending on the corresponding intervals of integration. These cases are as follows:

- Firstly, we approximate the solution on $[a,c]$ as follows:

Let $y''(x) = f(x)$ and define the residual function, $s(x)$, as $s(x) = y''(x) - f(x)$. Then the second approximate solution can be written as

$$y_2^1 = \alpha_1 + A(x-a) + \mathrm{c}_1(x-a)^2, \tag{3.3}$$

where value of A is determined later by using the continuity conditions of Eq. (3.1).

However, to find the value of coefficient c_1 in Eq. (3.3), we will differentiate it two times with respect to x and then substitute it in $s(x)$, to get

$$s(x) = 2c_1 - f(x). \tag{3.4}$$

Substitute $x = a$ in $s(x)$, we obtain $2c_1 - f(a) = 0$ or $c_1 = \frac{f(a)}{2}$. Thus, the second approximate solution must be

$$y_2^1 = \alpha_1 + A(x-a) + \frac{f(a)}{2}(x-a)^2. \tag{3.5}$$

To obtain the third approximation we define

$$y_3^1 = \alpha_1 + A(x-a) + c_1(x-a)^2 + c_2(x-a)^3. \tag{3.6}$$

Now, to find the value of coefficient c_2 in Eq. (3.6), we differentiate it two times with respect to x and then substitute it in $s'(x)$. Here, $(y_3^1)'' = 2c_1 + 6c_2(x-a)$, and $s'(x) = 6c_2 - f'(x)$. Substituting $x = a$ in $s'(x)$, we obtain $6c_2 - f'(a) = 0$ or $c_2 = \frac{f'(a)}{6}$. Consequently, the third approximate solution will be

$$y_3^1 = \alpha_1 + A(x-a) + \frac{f(a)}{2}(x-a)^2 + \frac{f'(a)}{6}(x-a)^3. \tag{3.7}$$

Similarly, to find out the fourth approximation we construct y_4^1 as follows:

$$y_4^1 = \alpha_1 + A(x-a) + c_1(x-a)^2 + c_2(x-a)^3 + c_3(x-a)^4. \tag{3.8}$$

Now, to find the value of coefficient c_3 in Eq. (3.8), we will differentiate it two times with respect to x and then substitute it in $s''(x)$. Here, $(y_4^1)'' = 2c_1 + 6c_2(x-a) + 12c_3(x-a)^2$ and $s''(x) = 24c_3 - f''(x)$. Substituting $x = a$ in $s''(x)$ we obtained $24c_2 - f''(a) = 0$ or $c_3 = \frac{f''(a)}{24}$. Therefore, the fourth approximation takes the form

$$y_4^1 = \alpha_1 + A(x-a) + \frac{f(a)}{2}(x-a)^2 + \frac{f'(a)}{6}(x-a)^3 + \frac{f''(a)}{24}(x-a)^4. \tag{3.9}$$

- Secondly, we approximate the solution on $[c, d]$ as follows:

Let $y'' = g(x)y(x) + f(x) + r$ and define the residual function $s(x) = y'' - g(x)y(x) + f(x) + r$. Then the second approximate solution can be written as

$$y_2^2 = B + C(x-c) + k_1(x-c)^2, \tag{3.10}$$

where values of B and C is determined later by using the continuity conditions of Eq. (3.1).

Now, to find the value of coefficient k_1 in Eq. (3.10), we will differentiate it two times with respect to x and then substitute it in $s(x)$, to get

$$s(x) = 2k_1 - g(x)[B + C(x-c) + k_1(x-c)^2] - f(x) - r. \tag{3.11}$$

Substitute $x = c$ in $s(x)$, we obtain $2k_1 - Bg(c) - f(c) - r = 0$ or $k_1 = \frac{Bg(c)+f(c)+r}{2}$. Therefore, the second approximation is

$$y_2^2 = B + C(x-c) + \frac{Bg(c)+f(c)+r}{2}(x-c)^2. \tag{3.12}$$

Again, the third approximation takes the form

$$y_3^2 = B + C(x-c) + k_1(x-c)^2 + k_2(x-c)^3. \tag{3.13}$$

To find the value of coefficient k_2 in Eq. (3.13), we will differentiate it two times with respect to x and then substitute it in $s'(x)$, to get,

$$\begin{aligned} s'(x) = 6k_2 - g(x)[C + 2k_1(x-c) + 3k_2(x-c)^2] - g'(x)[B + C(x-c) \\ + k_1(x-c)^2 + k_2(x-c)^3] - f'(x). \end{aligned} \tag{3.14}$$

Substitute $x = c$ in $s'(x)$, we obtained $6k_2 - Cg(c) - Bg'(c) - f'(c) = 0$ or $k_2 = \frac{Cg(c)+Bg'(c)+f\prime(c)}{6}$. Consequently, the third approximation takes the expansion form

$$\begin{aligned} y_3^2 = B + C(x-c) + \frac{Bg(c)+f(c)+r}{2}(x-c)^2 \\ + \frac{Cg(c)+Bg'(c)+f'^{(c)}}{6}(x-c)^3. \end{aligned} \tag{3.15}$$

To, find out the fourth approximation, we construct y_4^2 as follows:

$$y_4^2 = B + C(x-c) + k_1(x-c)^2 + k_2(x-c)^3 + k_3(x-c)^4. \tag{3.16}$$

To find out the value of coefficient k_3 in Eq. (3.16) we will differentiate it two times with respect to x and then substitute it in $s''(x)$, to get,

$$\begin{aligned} s''(x) = 24k_3 - g(x)[2k_1 + 6k_2(x-c) + 12k_3(x-c)^2] - 2g'(x)[B \\ + 2k_1(x-c) + 3k_2(x-c)^2 + 4k_3(x-c)^3] \\ - g''(x)[B + C(x-c) \\ + k_1(x-c)^2 + k_2(x-c)^3 + k_3(x-c)^4] - f''(x). \end{aligned} \tag{3.17}$$

Substitute $x = c$ in $s''(x)$, we obtain, $24k_3 - 2k_1g(c) - 2Cg'(c) - Bg''(c) - f''(c) = 0$, therefore $k_3 = \frac{2k_1g(c)+2Cg'(c)+Bg''(c)+f\prime(c)}{24}$. Hence,

$$y_4^2 = B + C(x-c) + \frac{Bg(c) + f(c) + r}{2}(x-c)^2 + \frac{Cg(c) + Bg'(c) + f'^{(c)}}{6}(x-c)^3 + \frac{2k_1 g(c) + 2Cg'(c) + Bg''(c) + f'^{(c)}}{24}(x-c)^4. \quad (3.18)$$

- Thirdly, we approximate the solution on $[d, b]$ as follows:

Let $y''(x) = f(x)$ and define the residual function $s(x) = y''(x) - f(x)$. Then the second approximate solution can be written as

$$y_2^3 = \alpha_2 + D(x-b) + r_1(x-b)^2, \quad (3.19)$$

where value of D is determined later by using the continuity conditions of Eq. (3.1).

To find the value of coefficient r_1 in Eq. (3.19) we will differentiate it two times with respect to x and then substitute it in $s(x)$. Here, $s(x) = 2r_1 - f(x)$. Substitute $x = b$ in $s(x)$, we obtained $2r_1 - f(b) = 0$ or $r_1 = \frac{f(b)}{2}$. Therefore, the second approximation is

$$y_2^3 = \alpha_2 + D(x-b) + \frac{f(b)}{2}(x-b)^2. \quad (3.20)$$

Again, to obtain the third approximation, we write

$$y_3^3 = \alpha_2 + D(x-b) + r_1(x-b)^2 + r_2(x-b)^3. \quad (3.21)$$

Now, to find the value of coefficient r_2 in Eq. (3.21) we will differentiate it two times with respect to x and then substitute it in $s'(x)$, to get, $s'(\mathrm{x}) = 6r_2 - f'(x)$. Substitute $x = b$ in $s'(x)$, we obtain $s'(\mathrm{b}) = 6r_2 - f'(\mathrm{b})$ or $r_2 = \frac{f'(b)}{6}$. Hence, the third approximation is

$$y_3^3 = \alpha_2 + D(x-b) + \frac{f(b)}{2}(x-b)^2 + \frac{f'(b)}{6}(x-b)^3. \quad (3.22)$$

Similarly, the fourth approximation can be constructed as follows:

$$y_4^3 = \alpha_2 + D(x-b) + r_1(x-b)^2 + r_2(x-b)^3 + r_3(x-b)^4. \quad (3.23)$$

So, to find the value of coefficient r_3 in Eq. (3.23) we will differentiate it two times with respect to x and then substitute it in $s''(x)$. Here, $s''(\mathrm{x}) = 24r_3 - f''(\mathrm{x})$. Anyhow, substitute $x = b$ in $s''(x)$, we obtain $24r_3 - f''(\mathrm{b}) = 0$ or $r_3 = \frac{f''(a)}{24}$, Thus, the fourth approximation will be

$$y_4^3 = \alpha_2 + D(x-b) + \frac{f(b)}{2}(x-b)^2 + \frac{f'(b)}{6}(x-b)^3 + \frac{f''(a)}{24}(x-b)^4. \tag{3.24}$$

In the next step of solution, we need to find out the values of parameters A, B, C,and D. In fact, this can be done by solving the following system of algebraic equations by using MATHCAD 14 software package:

$$\begin{aligned} &y_4^1(c) = y_4^2(c),\\ &y_4^2(d) = y_4^3(d),\\ &y'^1_4(c) = y'^2_4(c),\\ &y'^2_4(d) = y'^3_4(d). \end{aligned} \tag{3.25}$$

At the final step of our procedure, we write the following expansion which is the approximate solution by using four term approximations:

$$y''(x) = \begin{cases} y_4^1, a \le x \le c,\\ y_4^2, c \le x \le d,\\ y_4^3, d \le x \le b. \end{cases} \tag{3.26}$$

So, the approximate solutions for system of second order obstacle BVPs will be completely constructed. This procedure can be repeated till the arbitrary order coefficients of RPSM solutions are obtained.

4. Numerical and comparison results

To show behavior, efficiency, and applicability of the present RPSM, three obstacle BVPs will be solved approximately in this section.

Example 1: Consider the following system of second order obstacle BVP [4-8]:

$$y''(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{4},\\ y(x) - 1, & \frac{\pi}{4} \le x \le \frac{3\pi}{4},\\ 0, & \frac{3\pi}{4} \le x \le \pi, \end{cases} \tag{4.1}$$

subject to the boundary conditions

$$y(0) = 0, y(\pi) = 0, \tag{4.2}$$

where y and y' are continuous functions at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. Here, the exact solution is

$$y(x) = \begin{cases} \frac{4}{\gamma_1}x, & 0 \le x \le \frac{\pi}{4},\\ 1 - \frac{4}{\gamma_2}\cosh\left(\frac{\pi}{2} - x\right), & \frac{\pi}{4} \le x \le \frac{3\pi}{4},\\ \frac{4}{\gamma_1}(\pi - x), & \frac{3\pi}{4} \le x \le \pi. \end{cases} \tag{4.3}$$

where $\gamma_1 = \pi + 4\coth\left(\frac{\pi}{4}\right)$ and $\gamma_2 = \pi\sinh\frac{\pi}{4} + 4\coth\left(\frac{\pi}{4}\right)$.

The agreement between the exact and the numerical solutions is investigated for Example 1 at various x in the given interval by computing the numerical approximating of their exact solutions for the corresponding equivalent equations as shown in Tables 1. The computational results for the derivatives of the approximate solutions of Example 1 are tabulated in Table 2. While, numerical comparisons between the maximum absolute errors are tabulated next as given in Tables 3.

Table 1. Numerical results for Example 1.

x	$y(x)$	$y_{15}(x)$	$\lvert y_{15}(x) - y(x)\rvert$
0	0	0	0
$\pi/16$	0.08498999	0.08498999	$9.44076761\times 10^{-12}$
$3\pi/16$	0.25496996	0.25496996	$2.83223445\times 10^{-11}$
$\pi/4$	0.33995994	0.33995994	$3.77630704\times 10^{-11}$
$3\pi/8$	0.46279193	0.46279193	$6.00832162\times 10^{-11}$
$\pi/2$	0.50170955	0.50170955	$9.17877996\times 10^{-11}$
$5\pi/8$	0.46279193	0.46279193	$1.37415135\times 10^{-10}$
$3\pi/4$	0.33995994	0.33995994	$1.64360081\times 10^{-10}$
$13\pi/16$	0.25496996	0.25496996	$1.23269950\times 10^{-10}$
$15\pi/16$	0.08498999	0.08498999	$4.10899786\times 10^{-11}$
π	0	0	0

Table 2. The values of $\left|(y_{15})^{(i)}(x) - (y)^{(i)}(x)\right|$, $i = 0,1,2$ in Example 4.1.

x	$\lvert y_{15}(x) - y(x)\rvert$	$\lvert (y_{15})^{(1)}(x) - (y)^{(1)}(x)\rvert$	$\lvert (y_{15})^{(2)}(x) - (y)^{(2)}(x)\rvert$
0	0	$4.80810947\times 10^{-11}$	0
$\pi/16$	$9.44076761\times 10^{-12}$	$4.80800955\times 10^{-11}$	0
$3\pi/16$	$2.83223445\times 10^{-11}$	$4.80815388\times 10^{-11}$	0
$\pi/4$	$3.77630704\times 10^{-11}$	$4.80816498\times 10^{-11}$	0
$3\pi/8$	$6.00832162\times 10^{-11}$	$6.70509759\times 10^{-11}$	$6.01757533\times 10^{-11}$
$\pi/2$	$9.17877996\times 10^{-11}$	$9.64801426\times 10^{-11}$	$9.16273168\times 10^{-11}$
$5\pi/8$	$1.37415135\times 10^{-10}$	$1.35359363\times 10^{-10}$	$6.62333521\times 10^{-11}$
$3\pi/4$	$1.64360081\times 10^{-10}$	$2.09269657\times 10^{-10}$	0
$13\pi/16$	1.2326995×10^{-10}	$2.09269435\times 10^{-10}$	0
$15\pi/16$	$4.10899786\times 10^{-11}$	$2.09269546\times 10^{-10}$	0
π	0	$2.09269657\times 10^{-10}$	0

Table 3. Comparison of the maximum absolute errors for Example 1.

RPSM	Method in [24]	Method in [25]	Method in [26]	Method in [27]	Method in [28]
1.64×10^{-10}	4.46×10^{-10}	5.43×10^{-5}	8.43×10^{-5}	6.17×10^{-3}	4.04×10^{-3}

5. Conclusion

Building obstacle mathematical models for physical phenomenon, as well as developing numerical analytical solutions for such models are very important issue in mathematics, physics, and engineering. In this paper, a new analytic-numeric method, so-called RPSM, is proposed and applied to handle the second-order obstacle BVPs. Numerical results reveal the complete reliability and efficiency of the proposed method with a great potential in scientific applications.

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