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On the h-th Free Part of the Factorial

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

Abstract

Let $h \geq 2$ a fixed but arbitrary positive integer. Let $a_{n,h}$ be the h-th free part of $n!$. We prove the asymptotic formula $\log a_{n,h} \sim n \log h$. We also study the number of primes in the prime factorization of $n!$ with even exponent and odd exponent, and generalize these results. We prove that the number of odd exponents in the prime factorization of $n!$ is asymptotically equal to $(\log 2)\frac{n}{\log n}$ and the number of primes with even exponent in the prime factorization of $n!$ is asymptotically equal to $(1 - \log 2)\frac{n}{\log n}$. The Digamma function is used.

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1 Introduction

Let $h \geq 2$ a fixed but arbitrary positive integer. A positive integer is called h-th free when all exponents of the primes in its prime factorization are less than or equal to $h - 1$. The exponent of a prime p in the prime factorization of a certain positive integer will be denoted $e(p)$. For example, if $h = 2$ we obtain the square-free integers where all exponent $e(p) = 1$, if $h = 3$ we obtain the cubic-free integers where all exponent $e(p) \leq 2$, etc.

Let us consider a positive integer and a prime p with exponent $e(p)$ in its prime factorization, if we substitute the exponent $e(p)$ by the exponent $e'(p) \equiv e(p)(mod\ h)$ where $e'(p)$ takes values in the set $\{0, 1, 2, \ldots, h - 1\}$ and this is done with all prime p we obtain the h-th free part of the positive integer. For example, if $h = 3$ and the positive integer is $5^4.3^8.23^3.29^6$ then its cubic free part or 3-th free part is $5.3^2.23^0.29^0 = 5.3^2$. Clearly, The h-th free part of a positive integer is a h-th free number. Therefore all positive integer can be written in the unique way ab^h where a and b are positive integers and a is the h-th free part of the positive integer.

In particular if $h = 2$ then $n!$ can be written in this way, that is $n! = a_n b_n^2$ where a_n is the square free part of $n!$ and b_n is a positive integer. The following two formulae have been proved in [1].

$$\log a_n = n \log 2 + O(\sqrt{n}), \tag{1}$$

$$\log b_n = \frac{1}{2} n \log n - \frac{1 + \log 2}{2} n + O(\sqrt{n}). \tag{2}$$

In general given $h \geq 2$, $n!$ can be written in the unique way $n! = a_{n,h} b_{n,h}^h$ where $a_{n,h}$ denotes its h-th free part and $b_{n,h}$ is a positive integer. In this notation, hence, we have $a_{n,2} = a_n$ and $b_{n,2} = b_n$ (see (1) and (2)).

2 Main Results

In the following theorem we generalize (1) and (2).

Theorem 2.1 *If $n! = a_{n,h} (b_{n,h})^h$ then the following asymptotic formulae hold*

$$\log a_{n,h} = (\log h) n + o(n), \tag{3}$$

$$\log b_{n,h} = \frac{1}{h} n \log n - \frac{1 + \log h}{h} n + o(n). \tag{4}$$

Proof. We have the famous Euler's formula

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + o(1),$$

where γ is called Euler's constant.

If we put

$$\begin{aligned} A_k &= \left(\frac{1}{kh+1} - \frac{1}{kh+2} \right) + 2 \left(\frac{1}{kh+2} - \frac{1}{kh+3} \right) \\ &+ (h-1) \left(\frac{1}{kh+(h-1)} - \frac{1}{(k+1)h} \right) = \frac{1}{kh+1} + \frac{1}{kh+2} + \cdots \\ &+ \frac{1}{kh+(h-1)} - \frac{h-1}{(k+1)h} = \frac{1}{kh+1} + \frac{1}{kh+2} + \cdots + \frac{1}{(k+1)h} - \frac{1}{k+1} \end{aligned}$$

then we obtain by the Euler's formula

$$\begin{aligned}\sum_{k=0}^{s-1} A_k = \sum_{i=1}^{sh} \frac{1}{i} - \sum_{i=1}^{s} \frac{1}{i} &= (\log(sh) + \gamma + o(1)) - (\log s + \gamma + o(1)) \\ &= \log h + o(1).\end{aligned}$$

That is, the series of positive terms A_k converges and

$$\sum_{k=0}^{\infty} A_k = \log h. \tag{5}$$

We have (prime number theorem)

$$\vartheta(x) = \sum_{2 \le p \le x} \log p = x + o(x). \tag{6}$$

On the other hand, the exponent of the prime p in the prime factorization of $n!$ is (Legendre's theorem)

$$e(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor,$$

where $\lfloor . \rfloor$ denotes the integer part function. If p satisfies the inequality (where j is a fixed positive integer)

$$\frac{n}{j+1} < p \le \frac{n}{j}, \tag{7}$$

and the inequality

$$p > \sqrt{n}, \tag{8}$$

then we have

$$e(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = j.$$

Note that if n is sufficiently large inequalities (7) and (8) are fulfilled since $\sqrt{n} < \frac{n}{j+1}$.

Let $\epsilon > 0$, we choose the fixed positive integer s such that the following two inequalities hold

$$0 \le \sum_{k=s}^{\infty} A_k \le \epsilon, \tag{9}$$

$$0 \le (h-1)\frac{1}{sh} \le \epsilon. \tag{10}$$

Now, we have (see (6) and (5))

$$\log a_{n,h} = \sum_{2\leq p\leq n} e'(p)\log p = \sum_{2\leq p\leq \frac{n}{sh}} e'(p)\log p + \sum_{k=0}^{s-1}\left(\sum_{\frac{n}{kh+2}<p\leq \frac{n}{kh+1}} \log p\right.$$

$$+ \left.\sum_{\frac{n}{kh+3}<p\leq \frac{n}{kh+2}} 2\log p + \cdots + \sum_{\frac{n}{(k+1)h}<p\leq \frac{n}{kh+(h-1)}} (h-1)\log p\right)$$

$$= \sum_{2\leq p\leq \frac{n}{sh}} e'(p)\log p + n\sum_{k=0}^{s-1} A_k + o(n) = \sum_{2\leq p\leq \frac{n}{sh}} e'(p)\log p + (\log h)n$$

$$- \quad n\sum_{k=s}^{\infty} A_k + o(1)n \tag{11}$$

Besides, we have (see (6) and (10))

$$0 \leq \sum_{2\leq p\leq \frac{n}{sh}} e'(p)\log p \leq \sum_{2\leq p\leq \frac{n}{sh}} (h-1)\log p = (h-1)\frac{n}{sh} + o(1)n \leq 2\epsilon n \tag{12}$$

Note that there exists n_0 such that if $n \geq n_0$ the two $o(1)$ (see equations (11) and (12)) satisfy $|o(1)| \leq \epsilon$. Consequently, equations (11), (12) and (9) give

$$\left|\frac{\log a_{n,h}}{n} - \log h\right| \leq 4\epsilon \qquad (n \geq n_0) \tag{13}$$

Therefore (3) is proved, since ϵ can be arbitrarily small. Equation (4) is an immediate consequence of (3) and the well-known formula $\log n! = n\log n - n + O(\log n)$. The theorem is proved.

Let us consider the prime number theorem in the form

$$\pi(x) = \sum_{p\leq x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right), \tag{14}$$

where (as usual) $\pi(x)$ denotes the prime counting function.

The primes that appear in the prime factorization of $n!$ are the primes not exceeding n, consequently their number $N(n!)$ will be

$$N(n!) = \pi(n) = \sum_{p\leq n} 1 = \frac{n}{\log n} + o\left(\frac{n}{\log n}\right).$$

Let $h \geq 2$ an arbitrary but fixed positive integer. The number of primes p in the prime factorization of $n!$ such that $e(p) \equiv r \ (mod\ h)$, where $r \in \{1, 2, \ldots, h\}$, will be denoted $N_{h,r}(n!)$. For example if $h = 2$ then $N_{2,1}(n!)$ is the number of primes in the prime factorization of $n!$ with odd exponent and $N_{2,2}(n!)$ is the number of primes in the prime factorization of $n!$ with even exponent.

We have the following theorem

Theorem 2.2 *The following asymptotic formula holds*

$$N_{h,r}(n!) = C_{h,r}\frac{n}{\log n} + o\left(\frac{n}{\log n}\right), \tag{15}$$

where the constant

$$C_{h,r} = \sum_{k=0}^{\infty}\left(\frac{1}{hk+r} - \frac{1}{hk+(r+1)}\right), \tag{16}$$

and

$$\sum_{r=1}^{h} C_{h,r} = 1. \tag{17}$$

Proof. The proof is the same as the proof of Theorem 2.1. In this case we use equation (14). The theorem is proved.

Remark 2.3 *Note that the n-th partial sum of the series (16) is the difference of two harmonic progressions, namely*

$$\sum_{k=0}^{n}\left(\frac{1}{hk+r} - \frac{1}{hk+(r+1)}\right) = \sum_{k=0}^{n}\frac{1}{hk+r} - \sum_{k=0}^{n}\frac{1}{hk+(r+1)}.$$

Note also that in the series (16) we can remove the brackets since the resultant series is a Leibniz's series.

Example 2.4 *If $h = 2$ and $r = 1$ the series (16) is*

$$C_{2,1} = \sum_{k=0}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

and consequently (see (17))

$$C_{2,2} = 1 - \log 2.$$

Therefore, the number of primes with odd exponent in the prime factorization of $n!$ is

$$N_{2,1}(n!) = (\log 2)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right),$$

and the number of primes with even exponent in the prime factorization of $n!$ is

$$N_{2,2}(n!) = (1 - \log 2)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right).$$

Let us consider the Gamma function $\Gamma(z)$. The Digamma function $\Psi(z)$ is defined in the following way (see, for example, [2])

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We have the formula,

$$\Psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right),$$

where γ is the Euler's constant. Also, we have the formula

$$\Psi(1+z) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z} \right),$$

since $\Psi(1+z) = \Psi(z) + \frac{1}{z}$.

Let $a > 0$ and $d > 0$. We have the following well-known formula for the harmonic progression (see https://brilliant.org/wiki/harmonic-progression/)

$$\sum_{k=0}^{n} \frac{1}{dk+a} = \frac{1}{d}\log(dn+a) + \frac{1}{a} - \frac{\log d}{d} - \frac{1}{d}\Psi\left(1 + \frac{a}{d}\right) + o(1). \tag{18}$$

Equation (18) and Remark 2.3 give us the following expression for the numbers $C_{h,r}$ in terms of the Digamma function (see (16))

$$C_{h,r} = \frac{1}{r(r+1)} + \frac{1}{h}\left(\Psi\left(1 + \frac{r+1}{h}\right) - \Psi\left(1 + \frac{r}{h}\right)\right).$$

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[2] J. Havil, *Gamma,* Princeton University Press, 2003.

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