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Logarithm of the Exponents in the Prime Factorization of the Factorial

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Abstract

In this note we study the sum $\sum_{2\leq p\leq n}\log E(p)$ where $E(p)$ is the exponent of the prime p in the prime factorization of $n!$, the sum $\sum_{2\leq p\leq n} H_{E(p)}$, where H_k denotes the k-th harmonic number and another sums. We also consider the generalized harmonic numbers $H_{k,m}$ of order m and obtain a strong connection between these sums and the Riemann zeta function.

Mathematics Subject Classification: 11A99, 11B99

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1 Introduction

Let us consider the prime factorization of a positive integer a

$$a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k},$$

where $p_1, p_2, \ldots, p_k$ are the different primes in the prime factorization and $s_1, s_2, \ldots, s_k$ are the exponents. The total number of prime factors in the prime factorization is denoted $\Omega(a)$ (see [1, chapter XXII]), that is,

$$\Omega(a) = s_1 + s_2 + \cdots + s_k.$$

Let us consider the factorial $n!$, the exponent of a prime p in its prime factorization will be denoted $E(p)$. Then we can write the prime factorization of $n!$ in the form

$$n! = \prod_{2 \leq p \leq n} p^{E(p)}$$

since, clearly, the primes that appear in the prime factorization of $n!$ are the primes not exceeding n. It is well-known the following asymptotic formula (see [1, chapter XXII])

$$\Omega(n!) = \sum_{2 \leq p \leq n} E(p) = n \log \log n + An + o(n), \tag{1}$$

where A is a constant. Note that $\Omega(n!) = \sum_{2 \leq a \leq n} \Omega(a)$.

In this note we obtain an asymptotic formula for the sequence (compare with (1))

$$L(n!) = \sum_{2 \leq p \leq n} \log E(p).$$

Let us consider the k-th harmonic number H_k, namely $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

We obtain an asymptotic formula for the sequence (compare with (1))

$$M(n!) = \sum_{2 \leq p \leq n} H_{E(p)}.$$

We also consider the k-th generalized harmonic number of order $m \geq 2$, $H_{k,m}$, namely $H_{k,m} = \sum_{i=1}^{k} \frac{1}{i^m}$ and we obtain an asymptotic formula for the sequence (compare with (1))

$$M_m(n!) = \sum_{2 \leq p \leq n} H_{E(p),m}.$$

where the Riemann zeta function appear.

Finally, we consider the sum

$$\sum_{2 \leq p \leq n} \frac{1}{(E(p) - 1)!}.$$

where the e number appear.

2 Main Results

Theorem 2.1 *We have the asymptotic formula*

$$L(n!) = \sum_{2 \leq p \leq n} \log E(p) = C \frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = C\pi(n) + o(\pi(n)), \tag{2}$$

where the constant C is

$$C = \sum_{j=2}^{\infty} \frac{\log j}{j(j+1)}. \tag{3}$$

Proof. Note that, if we put

$$A_j = \log j \left(\frac{1}{j} - \frac{1}{j+1} \right) = \frac{\log j}{j(j+1)}, \tag{4}$$

the series of positive terms A_j converges, that is

$$\sum_{j=1}^{\infty} A_j = C. \tag{5}$$

We have (prime number theorem)

$$\pi(x) = \sum_{2 \le p \le x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x} \right). \tag{6}$$

Another more precise well-known formula is

$$\pi(x) = \sum_{2 \le p \le x} 1 = \frac{x}{\log x} + f_1(x) \left(\frac{x}{\log^2 x} \right), \tag{7}$$

where $|f_1(x)| < M$.

We also need the following well-known formula

$$\vartheta(x) = \sum_{2 \le p \le x} \log p = x + f_2(x) \frac{x}{\log x}, \tag{8}$$

where $|f_2(x)| < M$.

On the other hand, the exponent $E(p)$ of the prime p in the prime factorization of $n!$ is (Legendre's theorem)

$$E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor, \tag{9}$$

where $\lfloor . \rfloor$ denotes the integer part function. If p satisfies the inequality (where j is a fixed positive integer)

$$\frac{n}{j+1} < p \le \frac{n}{j}, \tag{10}$$

and the inequality

$$p > \sqrt{n}, \tag{11}$$

then we have

$$E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = j. \tag{12}$$

Note that if n is sufficiently large inequalities (10) and (11) are fulfilled since $\sqrt{n} < \frac{n}{j+1}$.

Let $\epsilon > 0$, we choose the fixed positive integer s such that the following inequalities hold

$$0 \leq \sum_{j=s}^{\infty} A_j \leq \epsilon, \tag{13}$$

$$0 \leq \frac{\log s}{s} \leq \epsilon, \tag{14}$$

$$\frac{M}{s} \leq \epsilon. \tag{15}$$

Now, we have (see (4), (5), (6), (10) and (12))

$$\begin{aligned} L(n!) &= \sum_{2 \leq p \leq n} \log E(p) = \sum_{2 \leq p \leq \frac{n}{s}} \log E(p) + \sum_{j=1}^{s-1} \left(\sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} \log j \right) \\ &= \sum_{2 \leq p \leq \frac{n}{s}} \log E(p) + \frac{n}{\log n} \sum_{j=1}^{s-1} A_j + o\left(\frac{n}{\log n} \right) = \sum_{2 \leq p \leq \frac{n}{s}} \log E(p) + C \frac{n}{\log n} \\ &- \frac{n}{\log n} \sum_{j=s}^{\infty} A_j + o(1) \frac{n}{\log n}. \end{aligned} \tag{16}$$

Besides, we have (see (7), (8) and (12))

$$\begin{aligned} &\sum_{2 \leq p \leq \frac{n}{s}} \log E(p) = \sum_{2 \leq p \leq \frac{n}{s}} \log \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \right) \leq \sum_{2 \leq p \leq \frac{n}{s}} \log \left(\sum_{k=1}^{\infty} \frac{n}{p^k} \right) = \log n \pi \left(\frac{n}{s} \right) \\ - &\sum_{2 \leq p \leq \frac{n}{s}} \log(p-1) = \log n \pi \left(\frac{n}{s} \right) - \sum_{2 \leq p \leq \frac{n}{s}} \log p + \sum_{2 \leq p \leq \frac{n}{s}} \log \left(\frac{p}{p-1} \right) \\ \leq &\ \log n \pi \left(\frac{n}{s} \right) - \vartheta \left(\frac{n}{s} \right) + \pi \left(\frac{n}{s} \right), \end{aligned} \tag{17}$$

since $1 < \frac{p}{p-1} \leq 2 < e$.

Let us consider the function

$$f_3(n) = \frac{\log n}{\log \frac{n}{s}}. \tag{18}$$

Note that $lim_{n\to\infty} f_3(n) = 1$.

Therefore (see (8) and (18))

$$\vartheta\left(\frac{n}{s}\right) = \frac{n}{s} + f_2\left(\frac{n}{s}\right)\frac{1}{s}\frac{n}{\log n} f_3(n). \tag{19}$$

On the other hand, we have (see (7) and (18))

$$\begin{aligned}\pi\left(\frac{n}{s}\right) &= \frac{n}{s\log\left(\frac{n}{s}\right)} + f_1\left(\frac{n}{s}\right)\frac{n}{s\left(\log\left(\frac{n}{s}\right)\right)^2} = \frac{1}{s}\frac{n}{\log n}\frac{1}{1-\frac{\log s}{\log n}} \\
&+ f_1\left(\frac{n}{s}\right)\frac{1}{s}\frac{n}{\log^2 n}(f_3(n))^2 = \frac{1}{s}\frac{n}{\log n}\left(1 + \frac{\log s}{\log n} f_4(n)\right) \\
&+ f_1\left(\frac{n}{s}\right)\frac{1}{s}\frac{n}{\log^2 n}(f_3(n))^2 = \frac{1}{s}\frac{n}{\log n} + \frac{\log s}{s}\frac{n}{\log^2 n} f_4(n) \\
&+ f_1\left(\frac{n}{s}\right)\frac{1}{s}\frac{n}{\log^2 n}(f_3(n))^2 = \frac{1}{s}\frac{n}{\log n} f_5(n),\end{aligned} \tag{20}$$

where $lim_{n\to\infty} f_4(n) = 1$, $lim_{n\to\infty} f_5(n) = 1$ and we have used the formula $\frac{1}{1-x} = 1 + x(1+o(1))$ $(x \to 0)$.

Substituting (19) and (20) into (17) we obtain (see (14) and (15))

$$\begin{aligned}\sum_{2\leq p\leq \frac{n}{s}} \log E(p) &\leq \log n\pi\left(\frac{n}{s}\right) - \vartheta\left(\frac{n}{s}\right) + \pi\left(\frac{n}{s}\right) = f_4(n)\frac{\log s}{s}\frac{n}{\log n} \\
&+ f_1\left(\frac{n}{s}\right)(f_3(n))^2\frac{1}{s}\frac{n}{\log n} - f_2\left(\frac{n}{s}\right) f_3(n)\frac{1}{s}\frac{n}{\log n} + f_5(n)\frac{1}{s}\frac{n}{\log n} \\
&\leq |f_4(n)|\frac{\log s}{s}\frac{n}{\log n} + \left|f_1\left(\frac{n}{s}\right)\right|(|f_3(n)|)^2\frac{1}{s}\frac{n}{\log n} + \left|f_2\left(\frac{n}{s}\right)\right||f_3(n)|\frac{1}{s}\frac{n}{\log n} \\
&+ |f_5(n)|\frac{1}{s}\frac{n}{\log n} \leq \left(2\frac{\log s}{s} + 4\frac{M}{s} + 2\frac{M}{s} + 2\frac{1}{s}\right)\frac{n}{\log n} \leq 10\epsilon\frac{n}{\log n}\end{aligned} \tag{21}$$

Note that there exists n_0 such that if $n \geq n_0$ then $|f_i(n)| \leq 2$ $(i = 3, 4, 5)$, since $f_i(n) \to 1$ $(i = 3, 4, 5)$.

Finally, equations (16), (13) and (21) give

$$\left|\frac{L(n!)}{\frac{n}{\log n}} - C\right| \leq \frac{\sum_{2\leq p\leq \frac{n}{s}} \log E(p)}{\frac{n}{\log n}} + \sum_{j=s}^{\infty} A_j + |o(1)| \leq 12\epsilon \quad (n \geq n_0) \tag{22}$$

Note that there exists n_0 such that if $n \geq n_0$ the $o(1)$ (see equation (22)) satisfies $|o(1)| \leq \epsilon$.

Therefore (2) is proved, since ϵ can be arbitrarily small. The theorem is proved.

Theorem 2.2 *We have the asymptotic formula*

$$M(n!) = \sum_{2\le p\le n} H_{E(p)} = \frac{\pi^2}{6}\frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = \frac{\pi^2}{6}\pi(n) + o(\pi(n))$$

Proof. The proof is the same as the proof of Theorem 2.1. Note that

$$\sum_{j=1}^{\infty}\frac{H_j}{j(j+1)} = \sum_{j=1}^{\infty}\frac{1}{j(j+1)} + \frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{j(j+1)} + \frac{1}{3}\sum_{j=3}^{\infty}\frac{1}{j(j+1)} + \cdots$$
$$= \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2) = \frac{\pi^2}{6}$$

since

$$\sum_{j=s}^{\infty}\frac{1}{j(j+1)} = \sum_{j=s}^{\infty}\left(\frac{1}{j} - \frac{1}{j+1}\right) = \frac{1}{s}$$

Besides we have the inequality

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} < \int_1^k \frac{1}{x}\, dx = \log k \qquad (k \ge 2)$$

and consequently the inequality

$$H_k \le 1 + \log k \qquad (k \ge 1)$$

Therefore (see (17))

$$\sum_{2\le p\le \frac{n}{s}} H_{E(p)} \le \sum_{2\le p\le \frac{n}{s}} (1 + \log E(p)) \le \log n\pi\left(\frac{n}{s}\right) - \vartheta\left(\frac{n}{s}\right) + 2\pi\left(\frac{n}{s}\right)$$

The theorem is proved.

We have the following generalization of Theorem 2.2 where the generalized harmonic number replaces the harmonic number.

Theorem 2.3 *We have the asymptotic formula*

$$M_m(n!) = \sum_{2\le p\le n} H_{E(p),m} = \zeta(m+1)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = \zeta(m+1)\pi(n) + o(\pi(n))$$

Proof. The proof is the same as the proof of Theorem 2.1. Note that

$$\sum_{j=1}^{\infty}\frac{H_{j,m}}{j(j+1)} = \sum_{j=1}^{\infty}\frac{1}{j(j+1)} + \frac{1}{2^m}\sum_{j=2}^{\infty}\frac{1}{j(j+1)} + \frac{1}{3^m}\sum_{j=3}^{\infty}\frac{1}{j(j+1)} + \cdots$$
$$= \quad 1 + \frac{1}{2^{m+1}} + \frac{1}{3^{m+1}} + \cdots = \zeta(m+1)$$

Besides

$$\sum_{2 \leq p \leq \frac{n}{s}} H_{E(p),m} \leq \zeta(m+1) \sum_{2 \leq p \leq \frac{n}{s}} 1 = \zeta(m+1)\pi\left(\frac{n}{s}\right)$$

The theorem is proved.

Theorem 2.4 *We have the asymptotic formula*

$$\sum_{2 \leq p \leq n} \frac{1}{(E(p)-1)!} = (e-2)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = (e-2)\pi(n) + o(\pi(n))$$

Proof. The proof is the same as the proof of Theorem 2.1. The theorem is proved.

In the following theorem we prove that the contribution of the primes $\frac{n}{s} < p \leq n$ to $\Omega(n!)$ is negligible (see equation (1)). Besides the s-th harmonic number H_s appear.

Theorem 2.5 *Let $s \geq 2$ an arbitrary but fixed positive integer. We have the asymptotic formula*

$$\sum_{\frac{n}{s} < p \leq n} E(p) = (-1 + H_s)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right)$$

Proof. We have

$$\sum_{\frac{n}{s} < p \leq n} E(p) = \sum_{j=1}^{s-1} \sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} j = \left(\sum_{j=1}^{s-1} \frac{j}{j(j+1)}\right)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right)$$

$$= (-1 + H_s)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right)$$

The theorem is proved.

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References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1960.

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