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On the Kernel Function

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

Abstract

In this article we study some properties of the kernel o radical function and some similar functions. The methods used are very elementary.

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1 Notations and Preliminary Results

A square-free or quadratfrei number is a number without square factors, a product of different primes. The first few terms of the integer sequence of square-free numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \dots$$

Let us consider the prime factorization of a positive integer $n \geq 2$

$$n = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}, \tag{1}$$

where $p_1, p_2, \dots p_t$ are the different primes in the prime factorization.

We have the following two arithmetical functions

$$u(n) = p_1 p_2 \cdots p_t. \tag{2}$$

The function $u(n)$ is well-known, it has many names in the literature, the largest square-free divisor of n, the kernel of n, the radical of n, etc. This function this relationed to the famous ABC conjecture. There is many papers dedicated to this function.

$$v(n) = \frac{n}{u(n)} = p_1^{s_1-1} p_2^{s_2-1} \cdots p_t^{s_t-1}. \tag{3}$$

We call $v(n)$ the remainder of n. Note that $v(n) = 1$ if and only if n is a square-free. We also define

$$w(n) = (p_1 + 1)(p_2 + 1) \cdots (p_t + 1). \tag{4}$$

If $n = 1$ then we put $u(n) = v(n) = w(n) = 1$.

We shall need the following theorem (see ([1], chapter XXII))

Theorem 1.1 *Let c_n $(n \geq 1)$ a sequence of real numbers. Let us consider the function*

$$A(x) = \sum_{n \leq x} c_n.$$

Suppose that $f(x)$ has a continuous derivative $f'(x)$ on the interval $[1, \infty]$, then the following formula holds

$$\sum_{n \leq x} c_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\ dt.$$

Now, we can prove the following general theorem.

Theorem 1.2 *Let us consider a strictly increasing sequence of positive integers, we denote b a positive integer in this sequence. Let $A(x)$ be the number of positive integers in this sequence not exceeding x. That is $A(x) = \sum_{b \leq x} 1$. Suppose that $A(x) = \rho x + o(x)$, where ρ is a positive real number, that is, ρ is the positive density of these integers. Then*

$$\sum_{b \leq x} b^k = \frac{\rho}{k+1} x^{k+1} + o\left(x^{k+1}\right),$$

where k is an arbitrary but fixed positive integer.

Proof. We use Theorem 1.1 con $f(x) = x^k$. Therefore

$$\begin{aligned}
&\sum_{b \leq x} b^k \\
= \;& A(x) x^k - \int_1^x A(t) k t^{k-1}\ dt = (\rho x + o(x))\, x^k - \int_1^x (\rho t + o(t))\, k t^{k-1}\ dt \\
= \;& \rho x^{k+1} + o\left(x^{k+1}\right) - k\rho \int_1^x t^k\ dt + \int_1^x o\left(t^k\right)\ dt \\
= \;& \rho x^{k+1} + o\left(x^{k+1}\right) - \rho \frac{k}{k+1} x^{k+1} + o\left(\int_1^x t^k\ dt\right) \\
= \;& \frac{\rho}{k+1} x^{k+1} + o\left(x^{k+1}\right)
\end{aligned}$$

The theorem is proved.

We shall need the following theorems on the distribution of square-free numbers. In this note a square-free number will be denoted q_1.

Theorem 1.3 *Let $Q_1(x)$ be the number of squarefree not exceeding x, we have*

$$Q_1(x) = \sum_{q_1 \le x} 1 = \frac{6}{\pi^2} x + o(x).$$

That is, the squarefree have positive density $\frac{6}{\pi^2}$.

Proof. See either ([1], chapter XVIII) or (for an alternative simple proof) [2].

In this article a square-free multiple of the different and fixed primes $p_1, p_2, \ldots, p_t$, that is multiple of the square-free $p_1 p_2 \cdots p_t$, will be denoted $q_{p_1 p_2 \cdots p_t}$.

Theorem 1.4 *Let $Q_{p_1 p_2 \cdots p_t}(x)$ be the number of squarefree multiple of the different and fixed primes $p_1, p_2, \ldots, p_t$ not exceeding x, we have*

$$Q_{p_1 p_2 \cdots p_t}(x) = \sum_{q_{p_1 p_2 \cdots p_t} \le x} 1 = \frac{6}{\pi^2} \frac{1}{(p_1+1)(p_2+1)\cdots(p_t+1)} x + o(x).$$

That is, these squarefree have positive density $\frac{6}{\pi^2}\frac{1}{(p_1+1)(p_2+1)\cdots(p_t+1)}$.

Proof. See [3].

We also need the geometric power series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1,$$

and the following well known fact, if the series of positive terms $\sum_{k=1}^{\infty} a_k$ converges then the infinit product $\prod_{k=1}^{\infty}(1+a_k)$ also converges to a positive number.

2 Main Results

Theorem 2.1 *Let k be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n \le x} u(n)^k = \frac{6}{\pi^2} \frac{C_k}{k+1} x^{k+1} + o\left(x^{k+1}\right), \tag{5}$$

where

$$C_k = \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n^{k+1}}$$

Proof. Let us consider the prime factorization of a positive integer $a \geq 2$

$$a = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t},$$

where $p_1, p_2, \dots p_t$ are the different primes in the prime factorization of a. We put

$$a' = p_1 p_2 \cdots p_t$$

and

$$a'' = (p_1 + 1)(p_2 + 1) \cdots (p_t + 1)$$

If $a = 1$ then we put $a' = a'' = 1$.

Therefore we have (see Theorem 1.3, Theorem 1.4 and Theorem 1.2)

$$\sum_{q_{a'} \leq x} q_{a'}^k = \frac{6}{\pi^2} \frac{1}{a''} \frac{x^{k+1}}{k+1} + o\left(x^{k+1}\right). \tag{6}$$

Let us consider the set N of all positive integers n not exceeding x. Now, let us consider the set T_a of all positive integers n not exceeding x with the same remainder a, that is, $T_a = \{n : n \leq x, v(n) = a\}$. Note that if $a_1 \neq a_2$ we have $T_{a_1} \cap T_{a_2} = \phi$, that is, the sets T_{a_1} and T_{a_2} are disjoint. Suppose that A_x (depending of x) is the greatest remainder among the number in the set N. Then

$$\bigcup_{a=1}^{A_x} T_a = N.$$

Therefore, the sets T_a are a partition of the set N. Note that some T_a can be empty.

The set of the kernels of the numbers in the set T_a will be denoted S_a. Hence,

$$S_a = \left\{ q_{a'} : q_{a'} \leq \frac{x}{a} \right\} \tag{7}$$

The series $\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{k+1}}$ converges. Hence

$$\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{k+1}} = C_k. \tag{8}$$

We choose B such that

$$\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{k+1}} < \epsilon, \tag{9}$$

$$\frac{1}{(B+1)^k} < \epsilon. \tag{10}$$

Therefore, we have (see (7), (6) and (8))

$$\begin{aligned}
\sum_{n\leq x} u(n)^k &= \sum_{a=1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right) = \sum_{a=1}^{B}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right) + \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right)\\
&= \sum_{a=1}^{B}\left(\frac{1}{a''}\frac{6}{\pi^2}\frac{x^{k+1}}{(k+1)a^{k+1}}\right) + o\left(x^{k+1}\right) + \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right)\\
&= \frac{6}{\pi^2}\frac{x^{k+1}}{k+1}\left(\sum_{a=1}^{B}\frac{1}{a''}\frac{1}{a^{k+1}}\right) + o\left(x^{k+1}\right) + \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right)\\
&= \frac{6}{\pi^2}\frac{x^{k+1}}{k+1}C_k - \frac{6}{\pi^2}\frac{x^{k+1}}{k+1}\left(\sum_{a=B+1}^{\infty}\frac{1}{a''}\frac{1}{a^{k+1}}\right) + o(1)\frac{6}{\pi^2}\frac{x^{k+1}}{k+1}\\
&+ \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right) \qquad (11)
\end{aligned}$$

Equation (11) can be written in the form

$$\frac{\sum_{n\leq x} u(n)^k}{\frac{6}{\pi^2}\frac{x^{k+1}}{k+1}} - C_k = -\left(\sum_{a=B+1}^{\infty}\frac{1}{a''}\frac{1}{a^{k+1}}\right) + o(1) + \frac{\sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right)}{\frac{6}{\pi^2}\frac{x^{k+1}}{k+1}}. \qquad (12)$$

Note that (see (7))

$$\bigcup_{a=B+1}^{A_x} S_a = \bigcup_{a=B+1}^{A_x}\left\{q_{a'} : q_{a'} \leq \frac{x}{a}\right\} \subseteq \left\{q_1 : q_1 \leq \frac{x}{B+1}\right\}$$

Note also that a given $q_{a'}$ can be in various sets S_a. Now, $q_{a'}a \leq x$, therefore the number of sets S_a such that $q_{a'} \in S_a$ does not exceed the number of multiples of $q_{a'}$ not exceeding x, namely $\left\lfloor \frac{x}{q_{a'}} \right\rfloor \leq \frac{x}{q_{a'}}$.

Hence, we have (see (10) and Theorem 1.2)

$$\begin{aligned}
0 &\leq \sum_{a=B+1}^{A(x)}\left(\sum_{q_{a'}\leq \frac{x}{a}} q_{a'}^k\right) \leq \sum_{q_1\leq \frac{x}{B+1}} q_1^k \left\lfloor \frac{x}{q_1} \right\rfloor \leq x \sum_{q_1\leq \frac{x}{B+1}} q_1^{k-1}\\
&= x\left(\frac{6}{\pi^2}\frac{1}{k}\frac{x^k}{(B+1)^k} + o\left(x^k\right)\right) = \frac{6}{\pi^2}\frac{x^{k+1}}{k+1}\left(\frac{k+1}{k}\frac{1}{(B+1)^k} + o(1)\right)\\
&\leq \frac{6}{\pi^2}\frac{x^{k+1}}{k+1}3\epsilon \qquad (13)
\end{aligned}$$

We choose x_0 such that if $x \geq x_0$ then $|o(1)| < \epsilon$ in equation (13) and $|o(1)| < \epsilon$ in equation (12).

Equations (12), (9) and (13) give

$$\left| \frac{\sum_{n\le x} u(n)^k}{\frac{6}{\pi^2}\frac{x^{k+1}}{k+1}} - C_k \right| \le 5\epsilon.$$

Therefore, since ϵ is arbitrarily small, we have

$$\lim_{x\to\infty} \frac{\sum_{n\le x} u(n)^k}{\frac{6}{\pi^2}\frac{x^{k+1}}{k+1}} = C_k.$$

That is (5). The theorem is proved.

Theorem 2.2 *The following formula holds*

$$C_k = \prod_p \left(1 + \frac{1}{(p+1)(p^{k+1}-1)}\right) \qquad (k \ge 1),$$

where the notation $\prod_p$ *mean that the product runs on all positive primes* p.

Proof. We have (see Theorem 2.1)

$$C_k = \prod_p \left(1 + \frac{1}{(p+1)p^{k+1}} + \frac{1}{(p+1)(p^{k+1})^2} + \frac{1}{(p+1)(p^{k+1})^3} + \cdots\right)$$
$$= \prod_p \left(1 + \frac{1}{p+1}\left(\frac{1}{1-\frac{1}{p^{k+1}}} - 1\right)\right) = \prod_p \left(1 + \frac{1}{(p+1)(p^{k+1}-1)}\right).$$

The theorem is proved.

Theorem 2.3 *The following asymptotic formula holds*

$$\sum_{n\le x} \frac{u(n)^b}{v(n)^c} = \frac{6}{\pi^2}\frac{C_{b+c}}{b+1}x^{b+1} + o\left(x^{b+1}\right), \tag{14}$$

where b *is a nonnegative integer and* c *is a positive integer.*

Proof. We have (see (1), (2) and (3))

$$\sum_{n\le x} \frac{u(n)^b}{v(n)^c} = \sum_{n\le x} \frac{u(n)^b}{\frac{n^c}{u(n)^c}} = \sum_{n\le x} \frac{u(n)^{b+c}}{n^c}.$$

Now, We shall use Theorem 1.1. We have (see Theorem 2.1)

$$A(x) = \sum_{n\le x} u(n)^{b+c} = \frac{6}{\pi^2}\frac{C_{b+c}}{b+c+1}x^{b+c+1} + o\left(x^{b+c+1}\right)$$

If we put $f(x) = x^{-c}$, $f'(x) = -cx^{-c-1}$, then we obtain

$$\begin{aligned}
\sum_{n\le x} \frac{u(n)^{b+c}}{n^c} &= \frac{6}{\pi^2}\frac{C_{b+c}}{b+c+1}x^{b+1} + o\left(x^{b+1}\right) + \int_1^x c\frac{6}{\pi^2}\frac{C_{b+c}}{b+c+1}t^b\ dt \\
+ & \int_1^x o\left(t^b\right)\ dt = \frac{6}{\pi^2}\frac{C_{b+c}}{b+1}x^{b+1} + o\left(x^{b+1}\right) + o\left(\int_1^x t^b\ dt\right) \\
= & \frac{6}{\pi^2}\frac{C_{b+c}}{b+1}x^{b+1} + o\left(x^{b+1}\right)
\end{aligned}$$

That is (14). The theorem is proved.

Corollary 2.4 *The following asymptotic formula holds*

$$\sum_{n\le x} \frac{1}{v(n)^k} = \frac{6}{\pi^2}C_k x + o(x).$$

Proof. We put $b = 0$ and $c = k$ in Theorem 2.3. The corollary is proved.

Now, we obtain some information on the sum of the remainders,

$$\sum_{n\le x} v(n) = \sum_{n\le x} \frac{n}{u(n)}.$$

Theorem 2.5 *The following limits hold*

$$\lim_{x\to\infty} \frac{\sum_{n\le x} v(n)}{x} = \infty, \tag{15}$$

$$\lim_{x\to\infty} \frac{\sum_{n\le x} v(n)}{x^{1+\alpha}} = 0. \tag{16}$$

for all $\alpha > 0$.

Proof. Let us consider the prime factorization of a positive integer $a \ge 2$

$$a = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t},$$

where $p_1, p_2, \ldots p_t$ are the different primes in the prime factorization of a. We put

$$a' = p_1 p_2 \cdots p_t,$$

and

$$a'' = (p_1+1)(p_2+1)\cdots(p_t+1).$$

If $a = 1$ then we put $a' = a'' = 1$.

Therefore we have (see Theorem 1.3 and Theorem 1.4)

$$Q_{a'}(x) = \frac{6}{\pi^2}\frac{1}{a''}x + o\left(x\right). \tag{17}$$

Let us consider the set N of all positive integers n not exceeding x. Now, let us consider the set T_a of all positive integers n not exceeding x with the same remainder a, that is, $T_a = \{n : n \leq x, v(n) = a\}$. Note that if $a_1 \neq a_2$ we have $T_{a_1} \cap T_{a_2} = \phi$, that is, the sets T_{a_1} and T_{a_2} are disjoint. Suppose that A_x (depending of x) is the greatest remainder among the number in the set N. Then

$$\bigcup_{a=1}^{A_x} T_a = N$$

Therefore, the sets T_a are a partition of the set N. Note that some T_a can be empty.

The set of the kernels of the numbers in the set T_a will be denoted S_a. Hence,

$$S_a = \left\{q_{a'} : q_{a'} \leq \frac{x}{a}\right\}. \tag{18}$$

The series $\sum_{a=1}^{\infty}\frac{1}{a''}$ diverges, since there are infinite values of a with the same a''. We choose B such that

$$\sum_{a=1}^{B}\frac{1}{a''} > \frac{\pi^2}{6}2M \tag{19}$$

Therefore, we have (see (17), (18) and (19))

$$\begin{aligned}\sum_{n\leq x} v(n) &= \sum_{a=1}^{A_x} aQ_{a'}\left(\frac{x}{a}\right) \geq \sum_{a=1}^{B} aQ_{a'}\left(\frac{x}{a}\right) = \sum_{a=1}^{B} a\left(\frac{6}{\pi^2}\frac{1}{a''}\frac{x}{a} + o(x)\right) \\ &= \frac{6}{\pi^2}x\sum_{a=1}^{B}\frac{1}{a''} + o(x) \geq 2Mx + o(x) = (2M + o(1))x \geq Mx\end{aligned}$$

That is

$$\frac{\sum_{n\leq x} v(n)}{x} \geq M.$$

Therefore, equation (15) is proved, since M is arbitrarily large.

If p denotes a positive prime and $\alpha > 0$ then the series

$$\sum_{p}\frac{1}{p^{1+\alpha} - p}$$

converges, since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha} - n}$$

converges. Therefore the infinite product

$$\prod_{p} \left(1 + \frac{1}{p^{1+\alpha} - p}\right)$$

converges. Now

$$1 + \frac{1}{p^{1+\alpha} - p} = 1 + \frac{1}{pp^{\alpha}} + \frac{1}{pp^{2\alpha}} + \frac{1}{pp^{3\alpha}} + \cdots .$$

Therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{u(n)} \frac{1}{n^{\alpha}}$$

converges. That is, we have

$$\sum_{n=1}^{\infty} \frac{1}{u(n)} \frac{1}{n^{\alpha}} = \sum_{n=1}^{\infty} \frac{v(n)}{n^{1+\alpha}} = C_{\alpha}. \tag{20}$$

Therefore (see (20))

$$A(x) = \sum_{n \le x} \frac{v(n)}{n^{1+\alpha}} = C_{\alpha} + o(1).$$

If we put $f(x) = x^{1+\alpha}$, $f'(x) = (1+\alpha)x^{\alpha}$, then we obtain (Theorem 1.1)

$$\begin{aligned} \sum_{n \le x} v(n) &= C_{\alpha} x^{1+\alpha} + o\left(x^{1+\alpha}\right) - \int_1^x C_{\alpha}(1+\alpha)t^{\alpha}\ dt + \int_1^x o\left(t^{\alpha}\right)\ dt \\ &= o\left(x^{1+\alpha}\right) + o\left(\int_1^x t^{\alpha}\ dt\right) = o\left(x^{1+\alpha}\right) \end{aligned}$$

That is, limit (16). The theorem is proved.

Corollary 2.6 *The following limits holds*

$$\lim_{x \to \infty} \frac{\sum_{n \le x} v(n)}{\sum_{n \le x} u(n)} = 0.$$

Therefore, the sum of the remainders of the numbers n not exceeding x is arbitrarily small in comparation with the sum of the kernels of the numbers n not exceeding x.

If we wish consider only the not quadratfrei numbers then also

$$\lim_{x \to \infty} \frac{\sum_{n \le x} v(n) - \sum_{q_1 \le x} v(q_1)}{\sum_{n \le x} u(n) - \sum_{q_1 \le x} u(q_1)} = 0.$$

Proof. It is an immediate consequence of Theorem 2.1 and Theorem 2.5 . Note also that $\sum_{q_1 \leq x} v(q_1) = \sum_{q_1 \leq x} 1 = Q_1(x) = \frac{6}{\pi^2} x + o(x)$ (see Theorem 1.3) and $\sum_{q_1 \leq x} u(q_1) = \sum_{q_1 \leq x} q_1 = \frac{6}{\pi^2} \frac{x^2}{2} + o(x^2)$ (see Theorem 1.2). The corollary is proved.

Clearly $v(n) \leq n$ for all n. In the following theorem we obtain a more precise result.

Theorem 2.7 *Let us consider the first n values of i, that is, $i = 1, 2, \ldots, n$. Let n_0 be the number of values of i such that $v(i) \geq i^\alpha$. Then*

$$\lim_{n \to \infty} \frac{n_0}{n} = 0. \tag{21}$$

Therefore, if n_1 is the number of values of i such that $v(i) < i^\alpha$ then

$$\lim_{n \to \infty} \frac{n_1}{n} = 1,$$

since $n_0 + n_1 = n$. The number α is an arbitrary but fixed positive real number.

Proof. We have (see (16))

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} v(i)}{n^{1+\alpha}} = 0. \tag{22}$$

Suppose that limit (21) is not true. Therefore there exists $\epsilon > 0$ such that for infinite values of n we have

$$\frac{n_0}{n} > \epsilon.$$

Therefore

$$\sum_{i=1}^{n} v(i) \geq \sum_{i} v(i) \geq \sum_{i} i^\alpha \geq \sum_{i=1}^{n_0} i^\alpha \geq \int_1^{n_0} x^\alpha \, dx \geq \int_1^{\epsilon n} x^\alpha \, dx$$
$$= \frac{\epsilon^{\alpha+1}}{\alpha+1} n^{1+\alpha} - \frac{1}{\alpha+1} \geq \frac{1}{2} \frac{\epsilon^{\alpha+1}}{\alpha+1} n^{1+\alpha}.$$

That is, for infinite values of n we have

$$\frac{\sum_{i=1}^{n} v(i)}{n^{1+\alpha}} \geq \frac{1}{2} \frac{\epsilon^{\alpha+1}}{\alpha+1}. \tag{23}$$

Now, equations (22) and (23) are an evident contradiction. Hence, limit (21) holds. The theorem is proved.

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