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The Kernel of Powerful Numbers

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Abstract

We generalize a theorem of the author to h-ful numbers and, for example, we obtain an asymptotic formula for the sum of the kernels of h-ful numbers. The methods used are very elementary.

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1 Notation and Preliminary Results

A square-free number is a number without square factors, a product of different primes. The first few terms of the integer sequence of square-free numbers are

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, \dots$$

Let us consider the prime factorization of a positive integer $n \geq 2$

$$n = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t}$$

where $q_1, q_2, \dots q_t$ are the different primes in the prime factorization.

We have the following two arithmetical functions

$$u(n) = q_1 q_2 \cdots q_t$$

The arithmetical function $u(n)$ is well-known in the literature, it is called kernel of n, radical of n, etc. There are many papers dedicated to this arithmetical function.

$$v(n) = \frac{n}{u(n)} = q_1^{s_1-1} q_2^{s_2-1} \cdots q_t^{s_t-1}$$

We call $v(n)$ the remainder of n. Note that $v(n) = 1$ if and only if n is a square-free. We also define the arithmetical function

$$w(n) = (q_1 + 1)(q_2 + 1) \cdots (q_t + 1)$$

In a previous article [8] the author proves using very elementary methods the following theorem on the kernel function.

Theorem 1.1 *Let k be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n \leq x} u(n)^k = \frac{6}{\pi^2} \frac{C_k}{k+1} x^{k+1} + o\left(x^{k+1}\right) \tag{1}$$

where

$$C_k = \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n^{k+1}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{k+1} - 1}\right) \tag{2}$$

The case $k = 1$ was studied in [1](see also [2]) and a better error term is obtained. We have

$$\sum_{n \leq x} u(n) = Cx^2 + O\left(x^{3/2} \log x\right)$$

where

$$C = \frac{1}{2} \prod_p \left(1 - \frac{1}{p(p+1)}\right)$$

This value of C can be obtained from our formulae (see (1) and (2) with $k = 1$), since

$$\frac{1}{2} \frac{6}{\pi^2} C_1 = \frac{1}{2} \prod_p \left(\left(1 - \frac{1}{p^2}\right)\left(1 + \frac{1}{(p+1)(p^2 - 1)}\right)\right) = \frac{1}{2} \prod_p \left(1 - \frac{1}{p(p+1)}\right)$$

Note that Theorem 1.1 is also true if $k = 0$, in this case we obtain the trivial equation

$$\sum_{n \leq x} 1 = x + o(x)$$

since

$$C_0 = \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n} = \prod_p \left(1 + \frac{1}{(p+1)(p-1}\right) = \prod_p \left(\frac{1}{1 - \frac{1}{p^2}}\right) = \frac{\pi^2}{6} \tag{3}$$

Equation (3) was proved in a former article of the author [7].

In this article we generalize Theorem 1.1 to h-ful numbers. A h-ful number is a positive integer such that in its prime factorization all multiplicities of the different primes are greater than or equal to h. That is, the prime factorization of a h-ful number is of the form

$$n_h = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t} \tag{4}$$

where $q_1, q_2, \ldots q_t$ are the different primes in the prime factorization and $s_i \geq h$ $(i = 1, 2, \ldots, t)$. We denote a h-ful number n_h. If $h = 2$ then the 2-ful numbers n_2 are also called either squareful or powerful numbers. If $h = 1$ then the 1-ful numbers n_1 are the positive integers. Note that h is a positive integer.

Let us consider the h-ful number (4). The kernel of n_h is $u(n_h) = q_1 q_2 \cdots q_t$. We define the h-kernel of n_h as $u(n_h)^h = q_1^h q_2^h \cdots q_t^h$ and the h-remainder of n_h as $v_h(n_h) = \frac{n_h}{u(n_h)^h} = q_1^{s_1-h} q_2^{s_2-h} \cdots q_t^{s_t-h}$.

We shall need the following well-known lemma (see ([3], chapter XXII))

Lemma 1.2 *Let c_n $(n \geq 1)$ a sequence of real numbers. Let us consider the function*

$$A(x) = \sum_{n \leq x} c_n$$

Suppose that $f(x)$ has a continuous derivative $f'(x)$ on the interval $[1, \infty]$, then the following formula holds

$$\sum_{n \leq x} c_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\ dt$$

The following general theorem is well-known (see [8]).

Theorem 1.3 *Let us consider a strictly increasing sequence of positive integers, we denote b a positive integer in this sequence. Let $A(x)$ be the number of positive integers in this sequence not exceeding x. That is $A(x) = \sum_{b \leq x} 1$. Suppose that $A(x) = \rho x + o(x)$, where ρ is a positive real number, that is, ρ is the positive density of these integers. Then*

$$\sum_{b \leq x} b^k = \frac{\rho}{k+1} x^{k+1} + o\left(x^{k+1}\right)$$

where k is an arbitrary but fixed positive integer.

We shall need the following theorems on the distribution of square-free numbers. In this note a square-free number will be denoted q_1.

Theorem 1.4 *Let $Q_1(x)$ be the number of squarefree not exceeding x, we have*

$$Q_1(x) = \sum_{q_1 \leq x} 1 = \frac{6}{\pi^2} x + o(x)$$

That is, the squarefree have positive density $\frac{6}{\pi^2}$.

Proof. See either ([3], chapter XVIII) or (for an alternative simple proof) [5].

In this note a square-free multiple of the different and fixed primes $q_1, q_2, \ldots, q_t$, that is multiple of the square-free $q_1 q_2 \cdots q_t$, will be denoted $q_{q_1 q_2 \cdots q_t}$.

Theorem 1.5 *Let $Q_{q_1 q_2 \cdots q_t}(x)$ be the number of squarefree multiple of the different and fixed primes $q_1, q_2, \ldots, q_t$ not exceeding x, we have*

$$Q_{q_1 q_2 \cdots q_t}(x) = \sum_{q_{q_1 q_2 \cdots q_t} \leq x} 1 = \frac{6}{\pi^2} \frac{1}{(q_1+1)(q_2+1)\cdots(q_t+1)} x + o(x)$$

That is, these squarefree have positive density $\frac{6}{\pi^2} \frac{1}{(q_1+1)(q_2+1)\cdots(q_t+1)}$.

Proof. See [6].

Theorem 1.6 *If $\alpha > 0$ the following two series of positive terms are convergent*

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}}, \qquad \sum_{n=1}^{\infty} \frac{1}{u(n) n^{\alpha}}$$

and besides the following two equations hold

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{\alpha} - 1}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{u(n) n^{\alpha}} = \prod_p \left(1 + \frac{1}{p(p^{\alpha} - 1}\right)$$

where the notation $\prod_p$ mean that the product runs on all positive primes p.

Proof. We have

$$\sum_{n=1}^{\infty} \frac{1}{w(n) n^{\alpha}} = \prod_p \left(1 + \frac{1}{(p+1)p^{\alpha}} + \frac{1}{(p+1)(p^{\alpha})^2} + \frac{1}{(p+1)(p^{\alpha})^3} + \cdots\right)$$

$$= \prod_p \left(1 + \frac{1}{(p+1)p^{\alpha}} \left(\frac{1}{1 - \frac{1}{p^{\alpha}}}\right)\right) = \prod_p \left(1 + \frac{1}{(p+1)(p^{\alpha} - 1}\right)$$

Now, the product

$$\prod_p \left(1 + \frac{1}{(p+1)(p^\alpha - 1)}\right)$$

converges to a positive number, since the series of positive terms

$$\sum_p \frac{1}{(p+1)(p^\alpha - 1)}$$

clearly converges. The theorem is proved.

2 Main Results

Now, we establish our main theorem. Theorem 1.1 is a particular case of this theorem when $h = 1$.

Theorem 2.1 *Let h be an arbitrary but fixed positive integer and let k be an arbitrary but fixed nonnegative integer. The following asymptotic formula holds*

$$\sum_{n_h \leq x} u(n_h)^k = \frac{6}{\pi^2} \frac{C_{k,h}}{k+1} x^{\frac{k+1}{h}} + o\left(x^{\frac{k+1}{h}}\right) \tag{5}$$

where

$$C_{k,h} = \sum_{n=1}^{\infty} \frac{1}{w(n)} \frac{1}{n^{\frac{k+1}{h}}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{k+1}{h}} - 1)}\right) \tag{6}$$

Proof. Let us consider the prime factorization of a positive integer $a \geq 2$

$$a = q_1^{s_1} q_2^{s_2} \cdots q_t^{s_t}$$

where $q_1, q_2, \ldots q_t$ are the different primes in the prime factorization of a. We put

$$a' = q_1 q_2 \cdots q_t$$

and

$$a'' = (q_1 + 1)(q_2 + 1) \cdots (q_t + 1)$$

If $a = 1$ then we put $a' = a'' = 1$.

Therefore we have (see Theorem 1.3, Theorem 1.4 and Theorem 1.5)

$$\sum_{q_{a'} \leq x} q_{a'}^k = \frac{6}{\pi^2} \frac{1}{a''} \frac{x^{k+1}}{k+1} + o\left(x^{k+1}\right) \tag{7}$$

Let us consider the set H of all h-full numbers n_h not exceeding x. Now, let us consider the set T_a of all h-full numbers n_h not exceeding x with the same h-remainder a, that is, $T_a = \{n_h : n_h \leq x, v_h(n_h) = a\}$. Note that if $a_1 \neq a_2$ we have $T_{a_1} \cap T_{a_2} = \phi$, that is, the sets T_{a_1} and T_{a_2} are disjoint. Suppose that A_x (depending of x) is the greatest h-remainder among the numbers in the set H. Then

$$\bigcup_{a=1}^{A_x} T_a = H$$

Therefore, the sets T_a are a partition of the set H. Note that some T_a can be empty.

The set of the h-kernel of the numbers in the set T_a will be denoted S_a. Hence,

$$S_a = \left\{ q_{a'}^h : q_{a'}^h \leq \frac{x}{a} \right\} = \left\{ q_{a'}^h : q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}} \right\} \tag{8}$$

The series $\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}}$ converges (see Theorem 1.6). Hence

$$\sum_{a=1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}} = C_{k,h} \tag{9}$$

We choose B such that (see Theorem 1.6)

$$\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}} < \epsilon \tag{10}$$

and

$$\frac{2(k+1)\pi^2}{6} \sum_{a=B+1}^{\infty} \frac{1}{a' a^{\frac{k+1}{h}}} < \epsilon \tag{11}$$

Therefore, we have (see (7), (8) and (9))

$$\begin{aligned}
\sum_{n_h \leq x} u(n_h)^k &= \sum_{a=1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) = \sum_{a=1}^{B} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) \\
&+ \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) = \sum_{a=1}^{B} \left(\frac{1}{a''} \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{(k+1) a^{\frac{k+1}{h}}} \right) + o\left(x^{\frac{k+1}{h}} \right) \\
&+ \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \leq \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) = \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} \left(\sum_{a=1}^{B} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}} \right) + o\left(x^{\frac{k+1}{h}} \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \le \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) = \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} C_{k,h} - \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} \left(\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}} \right) \\
&+ o(1) \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} + \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \le \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) \qquad (12)
\end{aligned}
$$

Equation (12) can be written in the form

$$
\begin{aligned}
&\frac{\sum_{n_h \le x} u(n_h)^k}{\frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1}} - C_{k,h} = - \left(\sum_{a=B+1}^{\infty} \frac{1}{a''} \frac{1}{a^{\frac{k+1}{h}}} \right) + o(1) \\
&+ \frac{\sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \le \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right)}{\frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1}} \qquad (13)
\end{aligned}
$$

Note that

$$
1^k + 2^k + \cdots + n^k \le \int_0^n x^k \, dx + n^k = \frac{n^{k+1}}{k+1} + n^k \le 2n^{k+1} \qquad (14)
$$

where $k \ge 0$ and $n \ge 1$.

We have (see (14) and (11))

$$
\begin{aligned}
0 &\le \sum_{a=B+1}^{A(x)} \left(\sum_{q_{a'} \le \frac{x^{(1/h)}}{a^{(1/h)}}} q_{a'}^k \right) \le \sum_{a=B+1}^{A(x)} \left((a')^k \sum_{q_1 \le \frac{x^{(1/h)}}{a' a^{(1/h)}}} q_1^k \right) \\
&\le \sum_{a=B+1}^{A(x)} \left((a')^k \sum_{n \le \frac{x^{(1/h)}}{a' a^{(1/h)}}} n^k \right) \le \sum_{a=B+1}^{A(x)} \left(2(a')^k \left(\frac{x^{(1/h)}}{a' a^{(1/h)}} \right)^{k+1} \right) \\
&= 2x^{\frac{k+1}{h}} \sum_{a=B+1}^{A(x)} \frac{1}{a' a^{\frac{k+1}{h}}} \le \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} \frac{2(k+1)\pi^2}{6} \sum_{a=B+1}^{\infty} \frac{1}{a' a^{\frac{k+1}{h}}} \\
&\le \epsilon \frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1} \qquad (15)
\end{aligned}
$$

We choose x_0 such that if $x \ge x_0$ then $|o(1)| < \epsilon$ in equation (13). Equations (13), (10) and (15) give

$$
\left| \frac{\sum_{n_h \le x} u(n_h)^k}{\frac{6}{\pi^2} \frac{x^{\frac{k+1}{h}}}{k+1}} - C_{k,h} \right| \le 3\epsilon
$$

Therefore, since ϵ is arbitrarily small, we have

$$\lim_{x\to\infty}\frac{\sum_{n_h\le x}u(n_h)^k}{\frac{6}{\pi^2}\frac{x^{\frac{k+1}{h}}}{k+1}}=C_{k,h}$$

That is (5). The theorem is proved.

Let $A_h(x)$ be the number of h-ful numbers not exceeding x. If we put $k=0$ in equations (5) and (6) we obtain the following corollary.

Corollary 2.2 *Let h be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$A_h(x)=\sum_{n_h\le x}1=\frac{6}{\pi^2}C_{0,h}x^{\frac{1}{h}}+o\left(x^{\frac{1}{h}}\right) \tag{16}$$

where

$$C_{0,h}=\sum_{n=1}^{\infty}\frac{1}{w(n)}\frac{1}{n^{\frac{1}{h}}}=\prod_p\left(1+\frac{1}{(p+1)(p^{\frac{1}{h}}-1}\right) \tag{17}$$

Remark 2.3 *Corollary 2.2 was proved by the author in a former article [7] using other elementary methods. More precise asymptotic formulae (better error term) are obtained using not elementary methods (see, for example, [4]). If $h=2$ then it is well-known that the constant can be written in terms of the Riemann zeta function $\zeta(s)$, that is, the value of the constant is $\frac{\zeta(3/2)}{\zeta(3)}$. This can be obtained from our formulae (16) and (17), since*

$$\frac{6}{\pi^2}C_{0,2}=\prod_p\left(\left(1-\frac{1}{p^2}\right)\left(1+\frac{1}{(p+1)(p^{1/2}-1)}\right)\right)$$
$$=\prod_p\left(1-\frac{1}{p^2}+\frac{p^{1/2}+1}{p^2}\right)=\prod_p\left(1+\frac{1}{p^{3/2}}\right)=\prod_p\left(\frac{\frac{1}{1-p^{-3/2}}}{\frac{1}{1-p^{-3}}}\right)=\frac{\zeta(3/2)}{\zeta(3)}$$

If we put $k=1$ in equations (5) and (6) we obtain the sum of the kernels of the h-ful numbers n_h not exceeding x. In the following corollary we establish the result.

Corollary 2.4 *Let h be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n_h\le x}u(n_h)=\frac{6}{\pi^2}\frac{C_{1,h}}{2}x^{\frac{2}{h}}+o\left(x^{\frac{2}{h}}\right) \tag{18}$$

where

$$C_{1,h}=\sum_{n=1}^{\infty}\frac{1}{w(n)}\frac{1}{n^{\frac{2}{h}}}=\prod_p\left(1+\frac{1}{(p+1)(p^{\frac{2}{h}}-1}\right) \tag{19}$$

Remark 2.5 *If $h = 2$ then (see (18), (19) and (3)) the sum of the kernels of the squareful numbers is*

$$\sum_{n_2 \le x} u(n_2) = \frac{1}{2}x + o\,(x)$$

In general (see (5), (6) and (3))the sum of the $(h-1)$-th powers of the kernels of the h-ful numbers is

$$\sum_{n_h \le x} u(n_h)^{h-1} = \frac{1}{h}x + o\,(x)$$

If we put $k = h$ in equations (5) and (6) we obtain the sum of the h-kernels of the h-ful numbers n_h not exceeding x. In the following corollary we establish the result.

Corollary 2.6 *Let h be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n_h \le x} u(n_h)^h = \frac{6}{\pi^2}\frac{C_{h,h}}{h+1}x^{1+\frac{1}{h}} + o\left(x^{1+\frac{1}{h}}\right)$$

where

$$C_{h,h} = \sum_{n=1}^{\infty} \frac{1}{w(n)}\frac{1}{n^{1+\frac{1}{h}}} = \prod_p \left(1 + \frac{1}{(p+1)(p^{1+\frac{1}{h}} - 1)}\right)$$

Using Lemma 1.2 and Corollary 2.2 we obtain the following result.

Theorem 2.7 *Let h be an arbitrary but fixed positive integer. The following asymptotic formula holds.*

$$\sum_{n_h \le x} n_h = \frac{6}{\pi^2}\frac{C_{0,h}}{h+1}x^{1+\frac{1}{h}} + o\left(x^{1+\frac{1}{h}}\right)$$

Remark 2.8 *Note that (see Corollary 2.4) the sum of the kernels of the h-ful numbers is negligible in comparation with the sum $\sum_{n_h \le x} n_h$ (see Theorem 2.7). On the other hand, the sum of the h-kernels (see Corollary 2.6) is a positive fraction of the sum $\sum_{n_h \le x} n_h$ (see Theorem 2.7).*

Using Lemma 1.2, Remark 2.5 and Corollary 2.6 we obtain the following result.

Theorem 2.9 *The following asymptotic formulae hold*

$$\sum_{n_h \le x} \frac{u(n_h)^{h-1}}{n_h} = \frac{1}{h}\log x + o(\log x) \tag{20}$$

$$\sum_{n_h \le x} \frac{1}{v_h(n_h)} = \sum_{n_h \le x} \frac{u(n_h)^h}{n_h} = \frac{6}{\pi^2}C_{h,h}x^{1/h} + o\left(x^{1/h}\right) \tag{21}$$

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