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A Short Note on a Generalization of Pure Ideals in Commutative Semigroups

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Abstract

In this paper, we give a generalization of pure ideal in commutative semigroup. S denotes a commutative semigroup with a nonzero identity and J denotes a proper ideal of S. We define a quasi r-ideal as a proper ideal J of S if $rS \cap J = rJ$ for each regular elements in S. In addition to giving basic properties of quasi r-ideals we have investigated relations between quasi r-ideals and other classical ideals such as pure ideals, von Neumann regular ideals and r-ideals.

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1 Introduction

In this study, by a semigroup we mean a semigroup S is a commutative multiplicative semigroup with nonzero identity. If X, Y is an arbitrary nonempty subset of S, then XY is defined by $\{xy : x \in X, y \in Y\}$, this product is valid for any two ideals of S. Let a be an element of S. Then principal ideal generated by a is assigned as $\langle a \rangle = \{sa : s \in S\} = aS = Sa$. Also, we define a as a regular element if whenever $ann(a) = \{r \in S : ra = 0\} = \langle 0 \rangle$, otherwise we say a is a zero divisor. Since 0 is always a zero divisor, we call a is proper zero divisor if $a \neq 0$. Further, we denote the set of all regular elements in S by $r(S)$. Let R be a ring and I a proper ideal of R. Then I is called an r-ideal if whenever $ab \in I$ with $ann(a) = \langle 0 \rangle$, then $b \in I$ [6]. Also, this notion is studied in commutative semigroup in [3]. However, the concept of r-ideals in semigroup has some differences relative to r-ideals of rings. For instance, in any ring R, a proper ideal I is an r-ideal of R if and only if $rR \cap I = rI$ for all $r \in r(R)$. In example 1, we show this is not necessarily true for semigroups. With this motivation, we give the notion of quasi r-ideal. We define a proper ideal J of a semigroup S as a quasi r-ideal if the equality $rS \cap J = rJ$ is satisfied for all $r \in r(S)$. Among many results in this paper, we show that quasi r-ideal is a generalization of pure ideals and von Neumann regular ideals: we define an ideal J of S that a pure (von Neumann regular) ideal if for any $a \in J$, then $a = ax$ ($a = a^2x$) for some $x \in J$ [1] ([5]). Recall that for any element a of S, $\langle a \rangle$ is strongly principal ideal if $\langle a \rangle\langle b \rangle = \langle a \rangle\langle c \rangle \neq \langle 0 \rangle$ implies $\langle b \rangle = \langle c \rangle$ for each $b, c \in S$ [2]. Further a semigroup is called an r-semigroup if all principal ideals are strongly principal. In Remark 1, it is shown that every r-ideal is also a quasi r-ideal. However, in Proposition 5, it is proved that the converse is true in any semigroup S whose regular principal ideals are strongly principal. Also, we show that, in Corollary 7, pure ideals, r-ideals, quasi r-ideals and von Neumann regular ideals coincide in any r-semigroup without a proper zero divisor. Also with lemma 8 and Theorem 9 we characterize the all quasi r-ideals of cartesian product of semigroups. Finally, we investigate the quasi r-ideals in polynomial semigroups by Theorem 10.

2 On quasi r-ideals

Definition 1 *A proper ideal J of S is said to be an r-ideal if $ab \in J$ with $ann(a) = \langle 0 \rangle$, then $b \in J$ for each $a, b \in S$* [3].

Remark 1 *Suppose that J is an r-ideal of a semigroup S and $r \in r(S)$. Then it is easily seen that $rS \cap J = rJ$ for each $r \in r(S)$. If $y \in rS \cap J$, then $y = rs \in J$ for some $s \in S$. Since J is an r-ideal, we conclude that $s \in J$, this implies $y = rs \in rJ$. Thus we get $rS \cap J = rJ$ as $rJ \subseteq rS \cap J$ is always true. However, the equality $rS \cap J = rJ$ (for each $r \in r(S)$) does not ensure that J is an r-ideal, see the following example.*

Example 1 *Consider the closed interval $S = [a, b]$, where $a, b \in \mathbb{R}$. We shall* *define the multiplication on S as follows:*

$$xy = \min\{x, y\}; \text{ for each } x, y \in S.$$

Notice that S is a commutative semigroup with identity is b and zero element is a. Now, take the ideal $J = [a, \varepsilon]$, where $\varepsilon < b$. It is clear that all regular elements of S are $S - \{a\}$. Let $r \in r(S)$. We may assume that $a < r < b$. Then it follows $rS \cap J = [a, r] \cap [a, \varepsilon] = [a, \min\{\varepsilon, r\}] = rJ$. By the way, J is not an r-ideal.

Definition 2 *We call a proper ideal J of S that a quasi r-ideal if $rS \cap J = rJ$ holds for each regular element r of S.*

By above observations, we can see that every r-ideal is also a quasi r-ideal. But the converse is not true, see Example 1.

Proposition 1 *Let S be a semigroup and J a proper ideal of S. Then the following statements are equivalent:*

(i) J is a quasi r-ideal of S.

(ii) For all nonempty subset $X \subseteq r(S)$, $XS \cap J = XJ$.

(iii) For all ideal I of S with $I \subseteq r(S)$, $IS \cap J = IJ$.

Proof. $(i) \Rightarrow (ii)$: Suppose that J is a quasi r-ideal of S and $X \subseteq r(S)$. Let $y \in XS \cap J$. Then $y = xs \in J$ for some $x \in X$ and $s \in S$. Since $X \subseteq r(S)$, it follows that $ann(x) = \langle 0 \rangle$ and thus $y = xs \in xS \cap J = xJ$ by (i). Hence we have $y = xs \in xJ \subseteq XJ$, so that $XS \cap J = XJ$ for each nonempty subset $X \subseteq r(S)$.

$(ii) \Rightarrow (iii)$: It is straightforward.

$(iii) \Rightarrow (i)$: Suppose that $r \in r(S)$ and $y \in rS \cap J$. It is sufficient to take $I = \langle r \rangle$ to prove the result. ■

Corollary 2 *(i) For a given principal ideal $\langle a \rangle$ of S, $\langle a \rangle$ is a quasi r-ideal of S if and only if $\langle a \rangle \cap \langle r \rangle = \langle ar \rangle$ for each $r \in r(S)$.*

(ii) In a semigroup S with $r(S) = \{1\}$, then every proper ideal is a quasi r-ideal.

Proposition 3 *The union of arbitrary family of quasi r-ideals is also a quasi r-ideal.*

Proof. Suppose that $\{J_i\}_{i \in \Delta}$ is a family of quasi r-ideals and $J = \bigcup_{i \in \Delta} J_i$. To prove that J is a quasi r-ideal, take $r \in r(S)$ and $y \in rS \cap J$. Then we get $y = rs \in J$ for some $s \in S$. This implies $y = rs \in J_{i_0}$ for some $i_0 \in \Delta$. Since J_{i_0} is a quasi r-ideal and $y = rs \in rS \cap J_{i_0}$, we have $y = rs \in rJ_{i_0}$, so that $y = rs \in rJ$. Consequently, we get $rS \cap J = rJ$, as required. ■

Now, we give a result clarifies the relation between quasi r-ideals and pure (von Neumann regular) ideals.

Proposition 4 *Every pure ideals and von Neumann regular ideals are also a quasi r-ideal.*

Proof. We need only prove that any von Neumann regular ideal is also a quasi r-ideal since pure ideals are von Neumann regular. Let J be a von Neumann regular ideal of S and $y \in rS \cap J$ for $r \in S$ with $ann(r) = \langle 0 \rangle$. Then we have $y = rs \in J$ for some $s \in S$. As J is a von Neumann regular ideal we conclude that $y = rs = (rs)^2 x = r(rs^2 x)$ for some $x \in J$. This implies $y = r(rs^2 x) \in rJ$, and thus $rS \cap J = rJ$ for each $r \in r(S)$. Consequently, J is a quasi r-ideal of S. ■

Following example reserves that the converse of previous proposition need not be true.

Example 2 *Consider the semigroup $S = \{0, 1, a, b\}$ with the following diagram:*

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	0	0

Note that S is a commutative semigroup with nonzero identity and 1 is the only regular element of S. By Corollary 2 every proper ideal is a quasi r-ideal, especially $J = \{0, a\}$. Since $a^2 = 0$, it is easy to see that J is not a pure (in fact it is not a von Neumann regular) ideal.

Proposition 5 *Suppose that S is a semigroup whose all regular principal ideals $\langle r \rangle$ are strongly principal. Then for any ideal J of S is an r-ideal if and only if J is a quasi r-ideal.*

Proof. The "if" part follows from Remark 1. Now, we prove that "only if" part of the proposition. Let J be a quasi r-ideal and $ab \in J$ with $ann(a) = \langle 0 \rangle$. Note that $ab \in aS \cap J = aJ$ since J is a quasi r-ideal. Thus we conclude that $ab = aj$ for some $j \in J$. If $ab = 0$, then $b = 0 \in J$. Assume that $ab = aj \neq 0$. Since $\langle a \rangle$ is strongly principal and $\langle a \rangle \langle b \rangle = \langle a \rangle \langle j \rangle \neq \langle 0 \rangle$, we have $\langle b \rangle = \langle j \rangle \subseteq J$ and this implies $b \in J$. Consequently, J is an r-ideal of S. ■

Corollary 6 *Suppose that S is a semigroup without a proper zero divisor. Then followings are equivalent:*

(i) J is a pure ideal of S.

(ii) J is a quasi r-ideal of S.

Proof. $(i) \Rightarrow (ii)$: It follows from Proposition 4.

$(ii) \Rightarrow (i)$: Suppose that J is a quasi r-ideal of S. To prove that J is a pure ideal, chose a nonzero $a \in J$. By assumption $a \in r(S)$, and thus $aS \cap J = aJ$ since J is a quasi r-ideal. This implies $a = ax$ for some $x \in J$, so that J is a pure ideal. ■

Recall that a semigroup S is called an r-semigroup if all principal ideals are strongly principal [2]. As an immediate consequences of Corollary 6, Proposition 4 and Proposition 5 we give the following result. Since its proof is clear, we omit the proof.

Corollary 7 *Let S be an r-semigroup without a proper zero divisor. Then the followings are equivalent:*

(i) J is an r-ideal;

(ii) J is a quasi r-ideal;

(iii) J is a pure ideal;

(iv) J is a von Neumann regular ideal.

Let S_1, S_2 be two semigroups and $S = S_1 \times S_2$. Under the componentwise multiplication, S becomes a semigroup with identity $(1_{S_1}, 1_{S_2})$ and zero element $(0_{S_1}, 0_{S_2})$. In addition, the set of all regular elements of $S_1 \times S_2$, $r(S_1 \times S_2) = r(S_1) \times r(S_2)$.

Lemma 8 *Suppose that $S = S_1 \times S_2$ and $J = J_1 \times J_2$, where J_i is an ideal of semigroup S_i for $i = 1, 2$. Then the followings are equivalent:*

(i) J is a quasi r-ideal of S.

(ii) J_1 is a quasi r-ideal of S_1 and $J_2 = S_2$ or J_2 is a quasi r-ideal of S_2 and $J_1 = S_1$ or J_1, J_2 are quasi r-ideals of S_1, S_2 respectively.

Proof. Let $J = J_1 \times J_2$ be a quasi r-ideal of $S = S_1 \times S_2$. Assume that $J_1 = S_1$. To prove that J_2 is a quasi r-ideal, take $r_2 \in r(S_2)$ and $y \in r_2S_2 \cap J_2$ for $y \in S_2$. Then we have $y = r_2s_2 \in J_2$ for some $s_2 \in S_2$. Also note that $(1_{S_1}, r_2) \in r(S_1 \times S_2)$ and $(1_{S_1}, r_2)(0_{S_1}, s_2) \in (1_{S_1}, r_2)(S_1 \times S_2) \cap (J_1 \times J_2)$. Since $J_1 \times J_2$ is a quasi r-ideal of $S_1 \times S_2$, we get the result that $(1_{S_1}, r_2)(0_{S_1}, s_2) \in (1_{S_1}, r_2)(J_1 \times J_2) = J_1 \times r_2J_2$, so that $(0_{S_1}, r_2s_2) \in J_1 \times rJ_2$, i.e. , $y = r_2s_2 \in r_2J_2$. Accordingly, J_2 is a quasi r-ideal of S_2. In other cases, one can similarly verify that (ii) holds. For the converse, assume that J_i's are quasi r-ideals of S_i for $i = 1, 2$ and $J = J_1 \times J_2$. Let $(r_1, r_2) \in r(S_1 \times S_2)$ and $(y_1, y_2) \in (r_1, r_2)(S_1 \times S_2) \cap (J_1 \times J_2)$. Then we have $y_i \in r_iS_i \cap J_i$. Since J_i is a quasi r-ideal of S_i we get $y_i \in r_iJ_i$, and thus $(y_1, y_2) \in r_1J_1 \times r_2J_2 = (r_1, r_2)(J_1 \times J_2)$, and this completes the proof. ■

Theorem 9 *Let $S = S_1 \times S_2 \times ... \times S_n$, where $n \geq 2$ and S_i's are semigroups. Suppose that $J = J_1 \times J_2 \times ... \times J_n$, where J_i's are ideals of S_i for $1 \leq i \leq n$. Then the following statements are equivalent:*

(i) J is a quasi r-ideal of S

(ii) I_i is a quasi r-ideal of S_i for some $i \in \{k_1, ..., k_t\}$ and $I_i = S_i$ for each $i \in \{1, ..., n\} - \{k_1, ..., k_t\}$.

Proof. We use induction on n. If $n = 2$, the result follows from previous lemma. Assume that $n \geq 3$ and the result is true for each $k \leq n-1$. Let $S' = S_1 \times S_2 \times ... \times S_{n-1}$, $J' = J_1 \times J_2 \times ... \times J_{n-1}$. Then $J = J' \times J_n$ is a quasi r-ideal of $S = S' \times S_n$ if and only if J' is a quasi r-ideal of S' and $J_n = S_n$ or $J' = S'$ and J_n is a quasi r-ideal of S_n or J', J_n are quasi r-ideals of S' and S_n, respectively. Hence, by induction hypothesis, it follows that the result is true. ■

Here, $S[x]$ denotes the commutative semigroup $\{sx^k : s \in S, k \geq 0\}$ with the multiplication $(s_1x^k)(s_2x^n) = s_1s_2x^{k+n}$ for each $s_1, s_2 \in S$ and $k, n \geq 0$, where x is an indeterminate. We know that if J is an ideal of S, then $J[x] = \{sx^k : s \in J\ , k \geq 0\}$ is also an ideal of $S[x]$. Also, note that $r(S[x]) = r(S)[x] = \{rx^k : ann(r) = \langle 0 \rangle,\ k \geq 0\}$.

Theorem 10 *Let S be a semigroup and J an ideal of S. Then J is a quasi r-ideal of S if and only if $J[x]$ is a quasi r-ideal of $S[x]$*

Proof. Suppose that J is a quasi r-ideal of S. Take $rx^k \in r(S[x])$ and $y \in (rx^k)S[x] \cap J[x]$. Then we have $r \in r(S)$ and $y = (rx^k)(sx^i)$ for some $sx^i \in S[x]$. This implies $y = rsx^{k+i} \in J[x]$. Since $rs \in J$ and J is a quasi r-ideal of S, we conclude that $rs \in rJ$, that is $rs = rb$ for some $b \in J$. Thus we get $y = (rx^k)(sx^i) = rsx^{k+i} = (rx^k)(bx^i) \in (rx^k)J[x]$. For the converse, assume that $J[x]$ is a quasi r-ideal of $S[x]$ and $y \in rS \cap J$ for some $r \in r(S)$. Then we get $y = rs \in J$ for some $s \in S$. Thus we infer $(rx)(sx) = rsx^2 \in (rx)S[x] \cap J[x]$. As $rx \in r(S[x])$ and $J[x]$ is a quasi r-ideal, we get $rsx^2 \in rxJ[x]$, and thus $rsx^2 = (rx)(bx)$ for some $b \in J$. Thus we conclude that $rs = rb \in rJ$, this completes the proof. ■

Suppose that $x_1, x_2, ..., x_n$ are indeterminates and $S[x_1, x_2, ..., x_n]$ denotes the polynomial semigroups. Now, we give a more general result. Since the proof is similar to previous theorem, it is omitted.

Theorem 11 *Suppose that J is a proper ideal of S and $x_1, x_2, ..., x_n$ are indeterminates. Then the followings are equivalent:*

(i) J is a quasi r-ideal of S.

(ii) $J[x_1, x_2, ..., x_n]$ is a quasi r-ideal of $S[x_1, x_2, ..., x_n]$.

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