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# On Pseudo-Union Curves in a Hypersurface of a Weyl Space

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**Abstract**

In this paper, firstly we have obtained the differential equation of pseudo-union curves and then we have defined pseudo-union curves in $W_n$. Secondly, we have expressed pseudo-asymptotic curves and pseudo-geodesic curves in $W_n$. Finally, we have given relation among these curves and by means of this relation, necessary theorems have been expressed.

**Mathematics Subject Classification:** 53B25

**Keywords:** Weyl space, the pseudo-union curves

## 1 Introduction

A manifold with a conformal metric $g_{ij}$ and a symmetric connection $\nabla_k$ satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1}$$

is called a Weyl space, which will be denoted by $W_n(g_ij, T_k)$. The vector field $T_k$ is named the complementary vector field. Under a renormalization of the metric tensor $g_{ij}$ in the form

$$\widetilde{g_{ij}} = \lambda^2 g_{ij} \tag{2}$$

the complementary vector field $T_k$ is transformed by the law

$$\widetilde{T_k} = T_k + \partial_k ln\lambda \tag{3}$$

where $\lambda$ is a scalar function [3].

The coefficients $\Gamma^i_{kl}$ of the symmetric connection $\nabla_k$ are given by

$$\Gamma^i_{kl} = \left\{ {i \atop kl} \right\} - g^{im}\big(g_{mk}T_l + g_{ml}T_k - g_{kl}T_m\big). \tag{4}$$

If under the transformation (2), the quantity $A$ is changed according to the rule

$$\widetilde{A} = \lambda^p A \tag{5}$$

then $A$ is called a satellite of $g_{ij}$ with weight $\{p\}$.

The prolonged derivative and prolonged covariant derivative of $A$ are, respectively defined by [1, 2]

$$\dot{\partial}_k A = \partial_k A - pT_k A \tag{6}$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{7}$$

Let $W_n(g_{ij}, T_k)$ be $n$-dimensional Weyl space and $W_{n+1}(g_{ab}, T_c)$ be $(n+1)$-dimensional Weyl space $(i, j, k = 1, 2, .., n;\ a, b, c = 1, 2, .., (n+1))$. Let $x^a$ and $u^i$ be the coordinates of $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. The metrics of $W_n(g_{ij}, T_k)$ and $W_{n+1}(g_{ab}, T_c)$ are connected by the relations

$$g_{ij} = g_{ab} x^a_i x^b_j \tag{8}$$

where $x^a_i$ is the covariant derivative of $x^a$ with respect to $u^i$.

The prolonged covariant derivative with respect to $u^k$ and $x^c$ are $\dot{\nabla}_k$ and $\dot{\nabla}_c$, respectively. These are related by the conditions

$$\dot{\nabla}_k A = x^c_k \dot{\nabla}_c A \quad (k = 1, 2, ..., n;\ c = 1, 2, ..., n+1). \tag{9}$$

Let the normal vector field $n^a$ of $W_n(g_{ij}, T_k)$ be normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\{x^i_a, n^a\}$ and its reciprocal $\{x^i_a, n^a\}$ are connected by the relations [3]

$$n^a n_a = 1, \quad n_a x^a_i = 0, \quad n^a x^i_a = 0, \quad x^a_i x^j_a = \delta^j_i. \tag{10}$$

Since the weight of $x_i^a$ is $\{0\}$, the prolonged covariant derivative of $x_i^a$, relative to $u^k$, is given by [3]

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a, \tag{11}$$

where $w_{ik}$ are the coefficients of the second fundamental form of $W_n(g_{ij}, T_k)$.

On the other hand, it is easy to see that the prolonged covariant derivative of $n^a$ is given by

$$\dot{\nabla}_k n^a = -w_{kl} g^{il} x_i^a. \tag{12}$$

By means of (10), the prolonged covariant derivative of $x_a^j$ is found to be [8]

$$\dot{\nabla}_k x_a^j = \Omega_k^j n_a. \tag{13}$$

Let $\underset{r}{v}^i$ $(i, r = 1, 2, ..., n)$ be the contravariant components of the vector field $\underset{r}{v}$ in $W_n(g_{ij}, T_k)$. Suppose that the vector fields $\underset{r}{v}$ $(r = 1, 2, ..., n)$ are normalized by the conditions $g_{ij} \underset{r}{v}^i \underset{r}{v}^j = 1$.

The reciprocal vector fields $\overset{r}{v}$ are defined by the relations [6]

$$\underset{r}{v}^i \overset{r}{v}_j = \delta_j^i, \quad \underset{r}{v}^i \overset{s}{v}_i = \delta_r^s \quad \big(i, j, r, s = 1, 2, ..., n\big). \tag{14}$$

The prolonged covariant derivatives of the vector field $\underset{r}{v}$ and its reciprocal $\overset{r}{v}$ are, respectively, given by [7]

$$\dot{\nabla}_k \underset{r}{v}^i = \underset{r}{\overset{s}{T}}_k \underset{s}{v}^i, \quad \dot{\nabla}_k \overset{r}{v}_i = -\underset{s}{\overset{r}{T}}_k \overset{s}{v}_i. \tag{15}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be the contravariant components of the vector field $\underset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. Denoting the components $\overset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$ by $\overset{r}{v}_a$ and $\overset{r}{v}_i$ , we have [8]

$$\underset{r}{v}^a = x_i^a \underset{r}{v}^i, \quad \overset{r}{v}_a = x_a^i \overset{r}{v}_i. \tag{16}$$

If $\underset{rr}{K}$ is the normal curvature of $W_n(g_{ij}, T_k)$ in the direction of $\underset{r}{v}$, we have

$$\underset{rr}{K} = w_{ij} \underset{r}{v}^i \underset{r}{v}^j. \tag{17}$$

Since the weight of $w_{ij}$ is $\{1\}$ and that of $\underset{r}{v}^i$ is $\{-1\}$, $\underset{rr}{K}$ is a satellite of $g_{ij}$ of $\{-1\}$.

The quantities

$$\overset{p}{\underset{r}{\eta}} = \overset{p}{\underset{r}{T}}_k \underset{r}{v}^k \quad (r, p = 1, 2, ..., n) \tag{18}$$

are called the geodesic curvature of the lines of the net $(\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ relative to $W_n(g_{ij}, T_k)$ [7].

The vector fields

$$\underset{p}{c}^i = \overset{r}{\underset{p}{\eta}} \underset{r}{v}^i \quad (i, r, p = 1, 2, ..., n) \tag{19}$$

are called the geodesic vector fields of the net $(\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ relative to $W_n(g_{ij}, T_k)$ [7].

If the components of the geodesic vector fields relative to $W_{n+1}(g_{ab}, T_c)$ are denoted by $\underset{r}{\overline{c}}^a$, then we have [8]

$$\underset{r}{v}^c \dot{\nabla}_c \underset{r}{v}^a = \underset{r}{\overline{c}}^a = \left(w_{ik} \underset{r}{v}^i \underset{r}{v}^k\right) n^a + \underset{r}{c}^i x^a_i. \tag{20}$$

Since the net $(\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$ is orthogonal, we have [7]

$$\overset{r}{\underset{r}{T}}_k = 0, \quad \overset{p}{\underset{r}{T}}_k + \overset{r}{\underset{p}{T}}_k = 0 \quad (r \neq p). \tag{21}$$

## 2 Preliminaries

Let $C : x^i = x^i(s)$ $(i = 1, 2, ..., n)$ be a curve in $W_n$, where $s$ is its arc length and $\underset{1}{v}$ is the tangent vector field of $C$ at the point $P$.

Let $\lambda$ be a unit vector field in $W_{n+1}$ and $\lambda^a$ be the contravariant components of $\lambda$. $\lambda$ is a congruence of unit vector fields. It can be expressed as

$$\lambda^a = x^a_i w^i + z n^a \quad (a = 1, 2, ..., n + 1) \tag{22}$$

where $w^i$ are the contravariant components of the vector field $w$ with respect to $W_n$ and $z$ is a scalar. Since $g_{ab}\lambda^a\lambda^b = 1$ we have

$$g_{ij} w^i w^j + z^2 = 1 \tag{23}$$

or

$$z^2 = 1 - g_{ij} w^i w^j. \tag{24}$$

Let $N^a$ be the contravariant components of a unit vector field which satisfies the conditions: it is linearly dependent on $\lambda$ and $\underset{1}{v}$ and it is orthogonal to $\underset{1}{v}$ [4]. Hence

$$g_{ab}N^a N^b = 1 \quad (b = 1, 2, ..., n+1) \tag{25}$$

and

$$g_{ab}N^a \underset{1}{v}^b = 0. \tag{26}$$

On the other hand, we know that, the relation between $\underset{1}{\overline{c}}^a$ and $\underset{1}{c}^i$ is as follows:

$$\underset{1}{\overline{c}}^a = \underset{1}{c}^i x_i^a + (w_{ij}\underset{1}{v}^i \underset{1}{v}^j)n^a \quad (j = 1, 2, ..., n) \tag{27}$$

where $\underset{1}{\overline{c}}^a$ and $\underset{1}{c}^i$ are the geodesic vector fields with respect to $W_{n+1}$ and $W_n$, respectively; $n^a$ are the contravariant components of a unit vector field normal to $W_n$ and $w_{ij}$ are the coefficients of the second fundamental form of $W_n$.

Since $N^a$ is linearly dependent on $\lambda$ and $\underset{1}{v}$, it can be written as

$$N^a = \alpha\lambda^a + \beta\underset{1}{v}^a \tag{28}$$

where $\alpha$ and $\beta$ are scalars.

Multiplying (28) by $g_{ab}N^b$, we get

$$g_{ab}N^a N^b = 1 = \alpha g_{ab}\lambda^a N^b \tag{29}$$

where $g_{ab}\underset{1}{v}^a N^b = 0$, or

$$\alpha = \frac{1}{g_{ab}\lambda^a N^b}. \tag{30}$$

Multiplying (28) by $g_{ab}\underset{1}{v}^b$, we obtain

$$g_{ab}N^a \underset{1}{v}^b = 0 = \alpha g_{ab}\lambda^a \underset{1}{v}^b + \beta \tag{31}$$

where $g_{ab}\underset{1}{v}^a \underset{1}{v}^b$, or

$$\beta = -\frac{\alpha g_{ab}\lambda^a \underset{1}{v}^b}{\alpha g_{ab}\lambda^a N^b}. \tag{32}$$

From (28), (30) and (32), we have

$$N^a = \frac{1}{g_{cd}\lambda^c N^d}\lambda^a - \frac{g_{cd}\lambda^c \underset{1}{v}^d}{g_{cd}\lambda^c N^d}\underset{1}{v}^a \quad (c, d = 1, 2, ..., n+1) \tag{33}$$

or

$$N^a = \frac{\left(x_i^a w^i + zn^a\right) - g_{cd}\left(x_j^c w^j + zn^c\right) \underset{1}{v}^k x_k^d \underset{1}{v}^a}{g_{cd}\left(x_j^c w^j + zn^c\right) N^d} \tag{34}$$

or

$$N^a = \frac{x_i^a w^i + zn^a - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^a}{z g_{cd} n^c N^d} \tag{35}$$

where $\underset{1}{v}^d = \underset{1}{v}^k x_k^d$, $g_{cd} x_j^c x_k^d = g_{jk}$, $g_{cd} n^c x_k^d = 0$ and $g_{cd} x_j^c N^d = 0$.

Using $\underset{1}{v}^a = \underset{1}{v}^i x_i^a$ and (24), we get

$$N^a = \pm \frac{x_i^a \left(w^i - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^i\right) + zn^a}{\sqrt{1 - g_{ij} w^i w^j}\, g_{cd} n^c N^d} \tag{36}$$

or

$$N^a = \pm \frac{x_i^a \left(w^i - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^i\right) + zn^a}{\sqrt{1 - \delta_j^p g_{ip} w^i w^j}\, g_{cd} n^c N^d} \quad (p = 1, 2, ..., n) \tag{37}$$

where $g_{ij} = \delta_j^p g_{ip}$ or

$$N^a = \pm \frac{x_i^a \left(w^i - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^i\right) + zn^a}{\sqrt{1 - g^{pm} g_{jm} g_{ip} w^i w^j}\, g_{cd} n^c N^d} \quad (m = 1, 2, ..., n) \tag{38}$$

where $\delta_j^p = g^{pm} g_{jm}$, or

$$N^a = \pm \frac{x_i^a \left(w^i - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^i\right) + zn^a}{\sqrt{1 - \sum_{r=1}^n \underset{r}{v}^p \underset{r}{v}^m g_{jm} g_{ip} w^i w^j}\, g_{cd} n^c N^d} \tag{39}$$

where $g^{pm} = \sum_{r=1}^n \underset{1}{v}^p \underset{1}{v}^m$, or

$$N^a = \pm \frac{x_i^a \left(w^i - g_{jk} w^j \underset{1}{v}^k \underset{1}{v}^i\right) + zn^a}{\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j}}. \tag{40}$$

The plus sign in (40) is to be taken when $z > 0$, the minus sign when $z < 0$. Thus (40) will reduce to $N^a = n^a$, when $\lambda$ is linearly dependent on $\underset{1}{v}$ and $n^a$; that is $w^i = k\underset{1}{v}^i$, $k$ being any constant different from unity.

Using (40), we have

$$n^a = \frac{N^a}{|z|}\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j} + x_i^a \left( g_{jk} \frac{w^j}{|z|} \underset{1}{v}^k \underset{1}{v}^i - \frac{w^i}{|z|} \right) \tag{41}$$

or

$$n^a = \frac{N^a}{|z|}\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j} + x_i^a \left( g_{jk} \rho^j \underset{1}{v}^k \underset{1}{v}^i - \rho^i \right) \tag{42}$$

where $\rho^j = \frac{w^j}{|z|}$.

If we write (42) in (27), we get

$$\underset{1}{\overline{c}}^a = \underset{1}{c}^i x_i^a + \underset{11}{K} \left[ \frac{N^a}{|z|}\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j} + x_i^a \left( g_{jk} \rho^j \underset{1}{v}^k \underset{1}{v}^i - \rho^i \right) \right] \tag{43}$$

or

$$\underset{1}{\overline{c}}^a = x_i^a \left[ \underset{1}{c}^i + \underset{11}{K} g_{jk} \rho^j \underset{1}{v}^k \underset{1}{v}^i - \underset{11}{K} \rho^i \right] + \underset{11}{K} \frac{N^a}{|z|}\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j} \tag{44}$$

where $\underset{11}{K} = w_{ij} \underset{1}{v}^i \underset{1}{v}^j$ is the normal curvature of $C$.

From (44), we have

$$\underset{1}{\overline{c}}^a = x_i^a \overline{p}^i + \underset{11}{\overline{K}} N^a \tag{45}$$

where

$$\overline{p}^i = \underset{1}{c}^i + \underset{11}{K} g_{jk} \rho^j \underset{1}{v}^k \underset{1}{v}^i - \underset{11}{K} \rho^i \tag{46}$$

and

$$\underset{11}{\overline{K}} = \underset{11}{K} \frac{1}{|z|}\sqrt{1 - \underset{1}{v}^p \underset{1}{v}^m g_{jm} g_{ip} w^i w^j}. \tag{47}$$

From (45):

**Definition 2.1** *For convenience,* $\underset{1}{\overline{c}}^a$ *in (45) is called relative first curvature vector field of* $C$ *at* $P$ *in* $W_{n+1}$ *and* $\overline{p}^i$ *is called relative first curvature vector field of* $C$ *at* $P$ *in* $W_n$. $\overline{K}_g^2 = g_{ij} \overline{p}^i \overline{p}^j$ *is called relative first curvature of* $C$ *at* $P$ *in* $W_n$.

# 3 Pseudo Union Curves

**Definition 3.1** *The totally pseudo-geodesic surface is defined by* $\underset{1}{v}^a$ *and* $\underset{1}{\overline{c}}^a$.

Let $\mu$ be a unit vector field in the direction of the curve of congruence of curves, one curve of which passes through each point of $W_n$. $\mu^a$, in general, are not normal to $W_n$ and it can be specified by

$$\mu^a = t^i x_i^a + rN^a \tag{48}$$

where $t^i$ and $r$ are parameters [5].

Then we have

$$g_{ab}\mu^a\mu^b = 1 \tag{49}$$

and

$$g_{ab}x_i^a N^b = 0. \tag{50}$$

With the help of (48), (49) and (50), we get

$$1 = g_{ab}\mu^a\mu^b = g_{ab}\big(t^i x_i^a + rN^a\big)\big(t^j x_j^b + rN^b\big) = g_{ij}t^i t^j + r^2$$

or

$$g_{ij}t^i t^j = 1 - r^2 \tag{51}$$

where $g_{ab}x_i^a x_j^b = g_{ij}$ and $g_{ab}N^a N^b = 1$.

If the pseudo-geodesic in $W_{n+1}$ in the direction of the curve of the congruence with covariant components $\mu^a$ is to be a pseudo-geodesic of the totally pseudo-geodesic surface, then it is necessary that $\mu^a$ be a linear combination of $\underset{1}{v}^a$ and $\underset{1}{\overline{c}}^a$, therefore

$$\mu^a = \alpha \underset{1}{v}^a + \beta \underset{1}{\overline{c}}^a \tag{52}$$

where $\alpha$ and $\beta$ scalars.

From (45), (48) and (52), we get

$$t^i x_i^a + rN^a = \alpha x_i^a \underset{1}{v}^i + \beta\big(x_i^a \overline{p}^i + \underset{11}{\overline{K}} N^a\big) \tag{53}$$

where $\underset{1}{v}^a = x_i^a \underset{1}{v}^i$.

Multiplying (53) by $g_{ab}x_j^b$ and summing for $a$ and $b$, we obtain

$$g_{ij}t^i = \alpha g_{ij}\underset{1}{v}^i + \beta g_{ij}\overline{p}^i \tag{54}$$

where $g_{ab}x_i^a x_j^b = g_{ij}$ and $g_{ab}N^a x_j^b = 0$.

Multiplying (54) by $\underset{1}{v}^j$, we get

$$g_{ij}t^i\underset{1}{v}^j = \alpha + \beta g_{ij}\overline{p}^i\underset{1}{v}^j \tag{55}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$.

From (46), we obtain

$$\begin{aligned} g_{ij}p^i\underset{1}{v}^j &= g_{ij}\underset{1}{c}^i\underset{1}{v}^j + \underset{11}{K}g_{tk}\rho^t\underset{1}{v}^k g_{ij}\underset{1}{v}^i\underset{1}{v}^j - \underset{11}{K}g_{ij}\rho^i\underset{1}{v}^j \\ &= g_{ij}\underset{1}{\overset{p}{\eta}}\underset{p}{v}^i\underset{1}{v}^j + \underset{11}{K}g_{tk}\rho^t\underset{1}{v}^k - \underset{11}{K}g_{ij}\rho^i\underset{1}{v}^j \\ &= g_{ij}\underset{1}{\overset{p}{T}}_m\underset{1}{v}^m\underset{p}{v}^i\underset{1}{v}^j \qquad (p = 2, 3, ..., n) \end{aligned}$$

$$g_{ij}p^i\underset{1}{v}^j = 0 \tag{56}$$

where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$ and $g_{ij}\underset{p}{v}^i\underset{1}{v}^j = 0 \quad (p = 2, 3, ..., n)$.

Using (56) in (55), we have

$$\alpha = g_{ij}t^i\underset{1}{v}^j. \tag{57}$$

Multiplying (53) by $g_{ab}N^b$ and summing for $a$ and $b$, we get

$$r = \beta\overline{\underset{11}{K}} \quad or \quad \beta = \frac{r}{\overline{\underset{11}{K}}} \tag{58}$$

where $g_{ab}x_i^a N^b = 0$ and $g_{ab}N^a N^b = 1$.

Writing (57) and (58) in (54), we obtain

$$g_{ij}t^i = \left(g_{kh}t^k\underset{1}{v}^h\right)g_{ij}\underset{1}{v}^i + \frac{r}{\overline{\underset{11}{K}}}g_{ij}\overline{p}^i \quad (h = 1, 2, ..., n). \tag{59}$$

Multiplying (59) by $g^{jm}$ and summing for $j$, we obtain

$$\begin{aligned} \delta_i^m t^i &= \left(g_{kh}t^k\underset{1}{v}^h\right)\delta_i^m\underset{1}{v}^i + \frac{r}{\overline{\underset{11}{K}}}\delta_i^m\overline{p}^i \\ t^m &= \left(g_{kh}t^k\underset{1}{v}^h\right)\underset{1}{v}^m + \frac{r}{\overline{\underset{11}{K}}}\overline{p}^m \end{aligned} \tag{60}$$

where $g^{jm}g_{ij} = \delta_i^m$.

From (60), we have

$$\frac{t^m}{r} = \left(g_{kh}\frac{t^k}{r}\underset{1}{v}^h\right)\underset{1}{v}^m + \frac{1}{\overline{\underset{11}{K}}}\overline{p}^m \tag{61}$$

or

$$\ell^m = \Big(g_{kh}\ell^k \underset{1}{v}^h\Big)\underset{1}{v}^m + \frac{1}{\underset{11}{\overline{K}}}\overline{p}^m \tag{62}$$

where $\frac{t^m}{r} = \ell^m$, or

$$\begin{aligned} \overline{p}^m + \underset{11}{\overline{K}} g_{kh}\ell^k \underset{1}{v}^h \underset{1}{v}^m - \underset{11}{\overline{K}}\ell^m &= 0 \quad (m = 1, 2, ..., n) \\ \overline{p}^m + \underset{11}{\overline{K}}\Big(g_{kh}\ell^k \underset{1}{v}^h \underset{1}{v}^m - \ell^m\Big) &= 0. \end{aligned} \tag{63}$$

Equation (63) is the differential equation of the pseudo-union curves. The solutions of the $n$ equations (63) determine the pseudo-union curves in $W_n$ to that congruence.

Let us denote the left hand side of (63) by $\overline{\eta}^m$:

$$\overline{\eta}^m = \overline{p}^m - \underset{11}{\overline{K}}\Big(\ell^m - g_{kh}\ell^k \underset{1}{v}^h \underset{1}{v}^m\Big) = \overline{p}^m - \underset{11}{\overline{K}} v^m = 0. \tag{64}$$

where $v^m = \ell^m - g_{kh}\ell^k \underset{1}{v}^h \underset{1}{v}^m$.

$\overline{\eta}^m$ are called the contravariant components of the pseudo-union curvature vector field.

**Definition 3.2** *Pseudo-union curve is defined as a curve whose pseudo-union curvature vector field is a null vector field:* $\overline{\eta}^m = 0$.

If $\phi = \angle(\mu^a, N^b)$, then $cos\phi = g_{ab}\mu^a N^b$ where $g_{ab}\mu^a\mu^b = 1$ and $g_{ab}N^a N^b = 1$. Since $g_{ab}\mu^a N^b = r$, $cos\phi = r$ is obtained. Since $g_{ij}t^i t^j = 1 - r^2 = sin^2\phi$, we have $g_{ij}\ell^i\ell^j = g_{ij}\frac{t^i}{r}\frac{t^j}{r} = \frac{sin^2\phi}{cos^2\phi} = tan^2\phi$. If $\alpha = \angle(\underset{1}{v}^i, \ell^j)$, then $cos\alpha = \frac{g_{ij}\underset{1}{v}^i\ell^j}{\sqrt{g_{ij}\ell^i\ell^j}}$ or $cos\alpha tan\phi = g_{ij}\underset{1}{v}^i\ell^j$ where $g_{ij}\underset{1}{v}^i\underset{1}{v}^j = 1$. From $cos\alpha tan\phi = g_{ij}\underset{1}{v}^i\frac{t^j}{r}$, we have $cos\alpha cos\phi tan\phi = cos\alpha sin\phi = g_{ij}\underset{1}{v}^i t^j$.

The magnitude $\overline{K}_u$ of the vector field $\overline{\eta}^k$ is

$$\begin{aligned} \overline{K}_u^2 &= g_{kh}\overline{\eta}^k\overline{\eta}^h \\ &= g_{kh}\Big(\overline{p}^k - \underset{11}{\overline{K}}\ell^k + \underset{11}{\overline{K}} g_{im}\ell^i \underset{1}{v}^m \underset{1}{v}^k\Big)\Big(\overline{p}^h - \underset{11}{\overline{K}}\ell^h + \underset{11}{\overline{K}} g_{im}\ell^i \underset{1}{v}^m \underset{1}{v}^h\Big) \\ &= g_{kh}\overline{p}^k\overline{p}^h - 2\underset{11}{\overline{K}} g_{kh}\overline{p}^k\ell^h + \underset{11}{\overline{K}}^2 g_{kh}\ell^k\ell^h - \underset{11}{\overline{K}}^2\Big(g_{im}\ell^i \underset{1}{v}^m\Big)^2 \\ &= \overline{K}_g^2 - 2\underset{11}{\overline{K}} g_{kh}\overline{p}^k\ell^h + \underset{11}{\overline{K}}^2 tan^2\phi - \underset{11}{\overline{K}}^2 cos^2\alpha tan^2\phi \\ &= \overline{K}_g^2 - 2\underset{11}{\overline{K}} g_{kh}\overline{p}^k\ell^h + \underset{11}{\overline{K}}^2 sin^2\alpha tan^2\phi \end{aligned} \tag{65}$$

where $g_{kh}\overline{p}^k\underset{1}{v}^h = 0$, $g_{kh}\underset{1}{v}^k\underset{1}{v}^h = 1$ and $\overline{K}_g^2 = g_{kh}\overline{p}^k\overline{p}^h$.

Multiplying (54) by $\overline{p}^j$, we have

$$g_{ij}t^i\overline{p}^j = \beta\overline{K}_g^2 \tag{66}$$

where $g_{ij}\underset{1}{v}^i\overline{p}^j = 0$ and $\overline{K}_g^2 = g_{ij}\overline{p}^i\overline{p}^j$, or

$$\beta = \frac{g_{ij}t^i\overline{p}^j}{\overline{K}_g^2} \tag{67}$$

Writing (57) and (67) in (54), we get

$$g_{ij}t^i = \left(g_{kh}t^k\underset{1}{v}^h\right)g_{ij}\underset{1}{v}^i + \frac{g_{kh}t^k\overline{p}^h}{\overline{K}_g^2}g_{ij}\overline{p}^i. \tag{68}$$

Multiplying (68) by $t^j$ and summing for $i$ and $j$, we obtain

$$\begin{aligned} g_{ij}t^it^j =&\left(g_{kh}t^k\underset{1}{v}^h\right)^2 + \frac{\left(g_{kh}t^k\overline{p}^h\right)^2}{\overline{K}_g^2} \\ sin^2\phi =&cos^2\alpha sin^2\phi + \frac{\left(g_{kh}t^k\overline{p}^h\right)^2}{\overline{K}_g^2} \\ sin^2\phi sin^2\alpha\overline{K}_g^2 =&\left(g_{kh}t^k\overline{p}^h\right)^2 \\ sin\phi sin\alpha\overline{K}_g =&g_{kh}t^k\overline{p}^h. \end{aligned} \tag{69}$$

From (69), we have

$$tan\phi sin\alpha\overline{K}_g = g_{kh}\ell^k\overline{p}^h. \tag{70}$$

Using (70) in (65), we get

$$\begin{aligned} \overline{K}_u^2 =&\overline{K}_g^2 - 2\underset{11}{\overline{K}}tan\phi sin\alpha\overline{K}_g + \underset{11}{\overline{K}}^2 sin^2\alpha tan^2\phi \\ =&\left(\overline{K}_g - \underset{11}{\overline{K}}sin\alpha tan\phi\right)^2 \\ \overline{K}_u =&\overline{K}_g - \underset{11}{\overline{K}}sin\alpha tan\phi. \end{aligned} \tag{71}$$

**Definition 3.3**

- *If the curve $C$ is a pseudo-union curve then $\overline{K}_u = 0$.*
- *If the curve $C$ is a pseudo-asymptote curve then $\underset{11}{\overline{K}} = 0$.*
- *If the curve $C$ is a pseudo-geodesic curve then $\overline{K}_g = 0$.*

From (71) and Definition 3.3:

**Theorem 3.4** *If the curve $C$ has any two of the following properties it also has the third:*

- *it is a pseudo-union curve,*
- *it is a pseudo-asymptote curve,*
- *it is a pseudo-geodesic curve*

*provided that $v^m$ are not the components of a null vector field.*

If $\phi = 0$ or $\alpha = 0$ or $\underset{11}{\overline{K}} = 0$, we obtain $\overline{K}_u = \overline{K}_g$.
Hence:

**Theorem 3.5** *The necessary and sufficient condition for a pseudo-union curve to be pseudo-geodesic is one of the following*

- *it is a pseudo-asymptotic curve,*
- *the congruence consist of the normals,*
- *the direction of the tangent vector field to $C$ coincides with that of the vector field $\ell^k$.*

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