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Characteristic Value of r on The Equation $11^x \equiv r \pmod{100}$

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Abstract

The equation of $11^x \equiv r \pmod{100}$, where r, x is positive integer has some unique solutions of r. For modulo 100, the solution of r are $0 \leq r < 100$, where the tens of r is the unit value of x and the unit of r is 1. In this paper will discuss some of the characteristics of solutions that satisfy the r value to the modulo 100. However, a value of 11^x if calculated manually will be a long process. Therefore, this paper uses a binomial coefficient formula for calculate the value of 11^x.

Keywords: Modulo, Application Binomial Coefficients, Characteristic of Division Algorithm

1 Introduction

Two integers a and r are said to be congruent modulo b [3, 4, 8, 7], symbolized by

$$a \equiv r \pmod{b}, \tag{1}$$

where b divides the difference $a-r$ or $a-r = qb$ for some integer q. By division algorithm [8] obtained that

$$a = qb + r, \qquad 0 \leq r \leq b, \tag{2}$$

the integers q and r respectively are called the quotient and remainder in the division of a by b.

Urroz and Yebra [10] found that, for every pair of integers a, b there exists an integer $x \geq e(b) + 1$ such that

$$a^x \equiv x \pmod{b^n}$$

where b even and squarefree. Define that $e(b)$ is the highest power of a prime dividing b. However, this equation does not apply for all x. So, it takes some criteria to determine the value of x that satisfies.

Furthermore German[6] expand it to b is prime p, thus be form the equation $a^x \equiv x \pmod{p^k}$. In his paper obtained that for $p \mid a$, $a^x \equiv x \pmod{p^k}$ if only if $x \equiv 0 \pmod{p^k}$. Whereas, for $(p, a) = 1$, $a^x \equiv x \pmod{p^n}$ if only if $x \equiv x_k(p, n) \pmod{p^k(p-1)}$ for some $0 \leq n \leq p - 2$.

Based on the features of the equations obtained by Urozz and Yebra [10] and German[6], authors expand for $a = 11$ on modulo 10^2 and satisfy for every positive integer $x \geq 0$.

$$11^x \equiv r \pmod{100}.$$

The value of 11^x if calculated manually will be a long process, moreover for large value of x. On [5, 8] explained several features of Pascal numbers, one of which is to calculate the value 11^x can be determined by the $x-$row of the Pascal triangle. Therefore 11^x can be expressed in the form of binomial coefficients. Every element on Pascal's triangle can be expressed as $\binom{x}{l}$, for every non-negative integers $l \leq x$ [2, 1].

$$\binom{x}{l} = \frac{x!}{l!\,(x-l)!}$$

If $x = 0$ and $l = 0$, then $\binom{0}{0} = 1$. Moreover $\binom{x}{0} = 1 = \binom{x}{x}$. The row on the Pascal's triangle start from $x = 0$ and $l = 0$ s expressed in form $\binom{0}{0}$.

Figure 1: Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
\binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \cdots & \binom{x}{l-1} & \binom{x}{l} & \cdots & \binom{x}{x} & & &
\end{array}$$

Based on the concept of binomial coefficient it is found that for every positive integer $x \geq 0$, digit of 11^x is

$$11^x = \underline{\left[\binom{x}{0}\right]}\ \underline{\left[\binom{x}{1}\right]}\ \underline{\left[\binom{x}{2}\right]} \cdots \underline{\left[\binom{x}{x-2}\right]}\ \underline{\left[\binom{x}{x-1}\right]}\ \underline{\left[\binom{x}{x}\right]} \tag{3}$$

where $\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \ldots \binom{x}{x-2}, \binom{x}{x-1}, \binom{x}{x}$ are digit of r successive.

Furthermore, Munadi [9] find the application of binomial coefficient on the power of a two-digit integer. Given two non-negative integers a and b that forming a two digit number $\underline{[a]}\ \underline{[b]}$ where a as ten and b as unit. So that

$$\begin{aligned}
(\underline{[1]}\ \underline{[1]})^0 &= (10+1)^0 = 1 = \underline{\left[\binom{0}{0}\right]} \\
(\underline{[1]}\ \underline{[1]})^1 &= (10+1)^1 = 11 = \underline{[1]}\ \underline{[1]} = \underline{\left[\binom{1}{0}\right]}\ \underline{\left[\binom{1}{1}\right]} \\
(\underline{[1]}\ \underline{[1]})^2 &= (10+1)^2 = 100 + 2 \cdot 10 \cdot 1 + 1 = 100 + 20 + 1 = \underline{[1]}\ \underline{[2]}\ \underline{[1]} \\
&= \underline{\left[\binom{2}{0}\right]}\ \underline{\left[\binom{2}{1}\right]}\ \underline{\left[\binom{2}{2}\right]} \\
(\underline{[1]}\ \underline{[1]})^3 &= (10+1)^3 = 1000 + 3 \cdot 100 \cdot 1 + 3 \cdot 10 \cdot 1 + 1 \\
&= 1000 + 300 + 30 + 1 = \underline{[1]}\ \underline{[3]}\ \underline{[3]}\ \underline{[1]} = \underline{\left[\binom{3}{0}\right]}\ \underline{\left[\binom{3}{1}\right]}\ \underline{\left[\binom{3}{2}\right]}\ \underline{\left[\binom{3}{3}\right]} \\
&\vdots
\end{aligned}$$

Based on the statement expressed by [8] and application binomial coefficient by [9] obtained form power of 11 in combination is

$$11^x = \underline{\left[\binom{x}{0}\right]}\ \underline{\left[\binom{x}{1}\right]}\ \underline{\left[\binom{x}{2}\right]} \cdots \underline{\left[\binom{x}{x-2}\right]}\ \underline{\left[\binom{x}{x-1}\right]}\ \underline{\left[\binom{x}{x}\right]} \tag{4}$$

where $\underline{\left[\binom{x}{x}\right]}$ as units, $\underline{\left[\binom{x}{x-1}\right]}$ as tens, $\underline{\left[\binom{x}{x-2}\right]}$ as hundreds and so on. But, if the result of combination more than one digit so take the last digit and then the other digit summed to the left.

2 The Main Results

Based on equation (1), for $a = 11^x$ and $b = 100$ obtained that

$$11^x \equiv r \pmod{100}, \qquad 0 \leq r < 100 \tag{5}$$

On this Table 1, given the reminder of 11^x on modulo 100 for $0 \leq x \leq 19$.

Table 1: Value of r on The Equation $11^x \equiv r \pmod{100}$

Value of 11^x	Value of (r)	Value of 11^x	Value of (r)
11^0	1	11^{10}	1
11^1	11	11^{11}	11
11^2	21	11^{12}	21
11^3	31	11^{13}	31
11^4	41	11^{14}	41
11^5	51	11^{15}	51
11^6	61	11^{16}	61
11^7	71	11^{17}	71
11^8	81	11^{18}	81
11^9	91	11^{19}	91

On the Table 1 obtained that if $x = 0$ then $r = 1$, if $x = 1$ then $r = 11$, if $x = 2$ then $r = 21$ up to $x = 9$ obtained $r = 91$. So that, it is obtained that for integer $0 \leq x \leq 9$, satisfy

$$11^x \equiv \underline{[x]}\ \underline{[1]} \pmod{100}, \qquad 0 \leq r < 100$$

where $\underline{[x]}$ and $\underline{[1]}$ are successive digits. However for $x = 10$ obtained $r = 1$, for $x = 11$ obtained $r = 11$ and so on. It always repeats on $x = 0, 10, 20, 30, \ldots$. So, obtained value of $r = 1, 11, 21, 31, 41, 51, 61, 71, 81, 91$.

Based on the above obtained characteristic value of r described in Theorem 2.1.

Theorem 2.1 *For every $x, m, n \in Z$ where $x = 10m + n$, $m \geq 0$ and $0 \leq n \leq 9$, satisfy*

$$11^{10m+n} = \underline{[n]}\ \underline{[1]} \pmod{100}$$

where $\underline{[n]}$ and $\underline{[1]}$ are the values of tens and units of the remainder in the division of 11^{10m+n} by 100.

Proof Based on equation (2), for $a = 11^x$, $b = 100$ and $x \geq 0$, can be form

$$11^x = 100q + r, \qquad 0 \leq r < 100.$$

Since 100 is the smallest three digit number, then the value of r at most only has two digits are tens and units. The values of tens and units of r in equation (4) are $\underline{\left[\binom{x}{x-1}\right]}$ and $\underline{\binom{x}{x}}$, so

$$11^x \equiv \underline{\left[\binom{x}{x-1}\right]}\ \underline{\left[\binom{x}{x}\right]} \pmod{100}$$

$$11^{10m+n} \equiv \underline{\left[\binom{10m+n}{(10m+n)-1}\right]}\ \underline{\left[\binom{10m+n}{10m+n}\right]} \pmod{100}$$

Then it is known that $x = 10m + n$ where $m \geq 0$ and $0 \leq n \leq 9$ so it tens of r is

$$\underline{\left[\binom{10m+n}{(10m+n)-1}\right]} = \underline{[10m+n]}$$

For every integer $m \geq 0$ obtained

$$\begin{aligned}\underline{[10(0)+n]} &= \underline{[n]}\\ \underline{[10(1)+n]} &= \underline{[10+n]}\\ \underline{[10(2)+n]} &= \underline{[20+n]}\\ \underline{[10(3)+n]} &= \underline{[30+n]}\\ &\vdots\end{aligned}$$

Based on the description for $m = 0$, the tens value of r consists of one digit number is n, whereas for $m > 0$ consists of two digit number or more. By the explanation about the application of binomial coefficients in two digit numbers on units tribe, tens, hundreds and so on only consist of one digit number. Therefore, taken as the tens value of r is the last digit n, so

$$11^{10m+n} \equiv \underline{[n]}\ \underline{[1]} \pmod{100}$$

Furthermore, on the table 1 it also found that if $x = 0, 10, 20$ then $r = 1$ and if $x = 1, 11, 21$ then $r = 11$. Therefore, by the statement if it is related to Theorem 2.1 then obtained Corollary 2.2 and Corollary 2.3.

Corollary 2.2 *If* $x = 10m$*, for every integers* $m \geq 0$*, then*

$$11^{10m} \equiv 1 \pmod{100}$$

Proof Known that $x = 10m$ where $m \geq 0$, so $11^x = 11^{10m}$. By Theorem 2.1 obtained that

$$11^{10m} \equiv \underline{\left[\binom{10m}{10m-1}\right]}\ \underline{\left[\binom{10m}{10m}\right]} \pmod{100} \equiv \underline{[10m]}\ \underline{[1]} \pmod{100}$$

where $\underline{[10m]}, \underline{[1]}$ are tens and units digit of r successive. Then it is known that $x = 10m = 10m + 0$, so $n = 0$. Since $n = 0$ by Theorem 2.1 obtained that tens value of r is $\underline{[0]}$ and units value is $\underline{[1]}$.

$$11^{10m} \equiv \underline{[0]}\ \underline{[1]} \pmod{100} \equiv \underline{[1]} \pmod{100}$$

And then obtained that

$$11^{10m} \equiv 1 \pmod{100}$$

Furthermore for $x = 10m + 1$ also obtained unique remainder, such as explained on Corollary 2.3.

Corollary 2.3 *If $x = 10m + 1$, for every integers $m \geq 0$, then*

$$11^{10m+1} \equiv 11 \pmod{100}$$

Proof Known that $x = 10m + 1$ where $m \geq 0$, and $n = 1$ so $11^x = 11^{10m+1}$. By Theorem 2.1 obtained that

$$11^{10m+1} \equiv \underline{\left[\binom{10m+1}{10m}\right]}\ \underline{\left[\binom{10m+1}{10m+1}\right]} \pmod{100} \equiv \underline{[10m+1]}\ \underline{[1]} \pmod{100},$$

where $[10m + 1], [1]$ are digit of r successive. Furthermore known that $n = 1$, so by Theorem 2.1 obtained tens value of r is $\underline{[1]}$ and units value is $\underline{[1]}$

$$11^{10m+1} \equiv \underline{[1]}\ \underline{[1]} \pmod{100}$$

So obtained that

$$11^{10m+1} \equiv 11 \pmod{100}$$

Example 2.4 *Let $m = 3$, so $x = 10(3) + n$.*

$$11^{10(3)+n} \equiv \underline{[n]}\ \underline{[1]} \pmod{100}$$

For $n = 0$ obtained

$$11^{10(3)+0} = 11^{30+0} \equiv \underline{[0]}\ \underline{[1]} \pmod{100} \equiv \underline{[1]} \pmod{100}$$

So $11^{30} = 1 \pmod{100}$

For $n = 1$ obtained

$$11^{10(3)+1} = 11^{30+1} \equiv \underline{[1]}\ \underline{[1]} \pmod{100}$$

So $11^{31} = 11 \pmod{100}$, *where 1 is units value of* 31.

Then, let $n = 4$ so that

$$11^{10(3)+4} = 11^{30+4} \equiv \underline{[4]}\ \underline{[1]} \pmod{100}$$

So $11^{34} = 41 \pmod{100}$, *where 4 is units value of* 34.

3 Conclusion

In this paper, characteristics of r on the equation $11^x \equiv r \pmod{100}$ for $x = 10m + n$ where $x, m, n \in N$ and $m \geq 0, 0 \leq n \leq 9$ is $\underline{[n]}\ \underline{[1]}$ where $\underline{[n]}$ is tens value of r and $\underline{[1]}$ as units. Moreover n is units value of x. Then obtained $r = 1$ for $x = 10m$ and $r = 11$ for $x = 10m + 1$.

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