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On the Equations $y^2 - 10x^2 = 9$ and $z^2 - 17x^2 = 16$

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Abstract

Let k be an integer, A is a set $\{x_1, x_2, x_3, ..., x_n\}$ with n positive different integers. A set is called P_k, if $i, j \in \mathbb{N}$ and $i \neq j$, $x_i x_j + k$ is a square of an integer. The purpose of this article is to give another proof of the non-extendibility of the set $P_{-1} = \{1, 10, 17\}$.

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1. Introduction

The problem of extending P_k sets is an old one dating from the times of Diophantus [4]. The most famous result in this area is due to Baker and Davenport [1], who proved that the P_1 set $\{1, 3, 8, 120\}$ can not be extended. Dujella [5] has a vast amount of literature about this interesting problem.

Brown [2] proved the conjecture for the triples $\left\{n^2+1, (n+1)^2+1, (2n+1)^2+1\right\}$ where $n \neq 0 \pmod 4$ and for the triples $\{2, 2n^2+2n+1, 2n^2+6n+5\}$ where $n \equiv 1 (\bmod 4)$. Then Brown proved non-extendibility of triples $\{1, 2, 5\}$ and $\{17, 26, 68\}$.

Mohanty and Ramasamy [8] proved that the P_{-1} set $\{1,5,10\}$ has no extension. Kedlaya [7] gave a list of P_{-1} triples $\{1,2,145\}$, $\{1,2,4901\}$, $\{1,5,65\}$, $\{1,5,20737\}$, $\{1,10,17\}$ and $\{1,26,37\}$. While he was constructing the list, he used a general elemantary approach based on an idea of Cohn and the theory of the Pell equation.

Muriefah and Rashed [9] proved that the set $P_{-1}=\{1,5,442\}$ can not be extended.

Filipin [6] proved that there do not exist different positive integers $c,d>1$ such that the product of any two distinct elements of the set $\{1,10,c,d\}$ diminished by 1 is a perfect square. Peker, Dujella and Cenberci [10] showed that for an integer k such that $|k|\geq 2$, the $D(-2k+1)-$triple $\{1,k^2,k^2+2k-1\}$ can not be extended to a $D(-2k+1)-$quadruple.

In this paper, we give another proof of the non-extendibility of the P_{-1} set $\{1,10,17\}$ by using the same method with Muriefah and Rashed.

For proof, we need the following lemma from [3].

Lemma 1. *Let $u_n+v_n\sqrt{D}$ $(n=0,1,2,...)$ be the solution of $u^2-Dv^2=N$ in a fixed class k, where N is given nonzero integer, then*

(1) $u_{-r}=u_r$
(2) $v_{-r}=-v_r$
(3) $u_{r+s}=A_su_r+DB_sv_r$
(4) $v_{r+s}=B_su_r+A_sv_r$

where $A_s+B_s\sqrt{D}=\left(a+b\sqrt{D}\right)^s$, $s=1,2,3,...$ and $a+b\sqrt{D}$ is the fundamental solution of the Pell equation $A^2-DB^2=1$.

Now we can give our theorem.

Theorem 1. *The P_{-1} set $\{1,10,17\}$ can not be extended.*

Proof. Assume that there is a positive integer m and the set $P_{-1}=\{1,10,17\}$ can be extended with m. Then let us find integers x,y,z such that,

$$m-1 = x^2 \quad (1)$$
$$10m-1 = y^2 \quad (2)$$
$$17m-1 = z^2 \quad (3)$$

where x,y,z are some integers satisfying the above equations. Eliminating of m between (1) , (2) and (1) , (3) yields

$$y^2-10x^2 = 9 \quad (4)$$
$$z^2-17x^2 = 16 \quad (5)$$

respectively. So we have to obtain the solutions of the Pell equation (4) with the restriction given by (5).

The Pell equation $u^2 - 10v^2 = 1$ has the fundamental solution $u_1 = 19$, $v_1 = 6$. The equation (4) has six general solutions (6) (7) (8) (9) (10) (11) , but it is enough to analyze the first three equations (6) (7) (8) .

$$y_n + x_n\sqrt{10} = \left(7 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^n \quad (6)$$

$$y_n + x_n\sqrt{10} = \left(7 - 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^n \quad (7)$$

$$y_n + x_n\sqrt{10} = 3\left(19 + 6\sqrt{10}\right)^n \quad (8)$$

$$y_n + x_n\sqrt{10} = \left(-7 + 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^n \quad (9)$$

$$y_n + x_n\sqrt{10} = \left(7 - 2\sqrt{10}\right)\left(19 + 6\sqrt{10}\right)^n \quad (10)$$

$$y_n + x_n\sqrt{10} = -3\left(19 + 6\sqrt{10}\right)^n \quad (11)$$

respectively. Since $u_n + v_n\sqrt{10} = \left(19 + 6\sqrt{10}\right)^n$, we get the following table of values:

n	u_n	v_n
0	1	0
1	19	6
2	721	228
3	27379	8658
4	1039681	328776

Table 1

If we consider the equation (6) , we can calculate the following values:

n	y_n	x_n
0	7	2
1	253	80
2	9607	3038
3	364813	115364
4	13853287	4380794

Table 2

From the *Lemma* 1 we can write $x_{n+4} = 328776y_n + 1039681x_n$, so $x_{n+4} \equiv 53353x_n \pmod{328776}$ which implies $x_{n+4} \equiv x_n \pmod{19}$. Equation (5) becomes

$$z^2 \equiv 16 + 17x_n^2 \pmod{19}$$

We must think four different cases for this equation.

The first one is $n \equiv 0 \pmod 4$, then $x_n \equiv x_0 \equiv 2 \pmod{19}$. This implies $z^2 \equiv 8 \pmod{19}$, but the Legendre symbol $\left(\dfrac{8}{19}\right) = -1$.

The second case is $n \equiv 1 \pmod 4$, then $x_n \equiv x_1 \equiv 80 \pmod{19}$. This implies $z^2 \equiv 3 \pmod{19}$, but $\left(\dfrac{3}{19}\right) = -1$.

The third case is $n \equiv 2 \pmod 4$, then $x_n \equiv x_2 \equiv 3038 \equiv 17 \pmod{19}$. This implies $z^2 \equiv 8 \pmod{19}$ and this is the same result with the first case.

The last case is $n \equiv 3 \pmod 4$, then $x_n \equiv x_3 \equiv 115364 \equiv 15 \pmod{19}$. This implies $z^2 \equiv 3 \pmod{19}$ and this is the same result with the second case.

For the four cases, there exists no integer such that x_n simultaneously satisfies (4) and (5).

Now if we consider the equation (7), then one can check that the following table of values:

n	y_n	x_n
0	7	-2
1	13	4
2	487	154
3	18493	5848
4	702247	222070

Table 3

From the *Lemma* 1 we can write $x_{n+4} = 328776y_n + 1039681x_n$, so $x_{n+4} \equiv 53353x_n \pmod{328776}$ which implies $x_{n+4} \equiv x_n \pmod{19}$. Equation (5) becomes

$$z^2 \equiv 16 + 17x_n^2 \pmod{19}$$

We must check four different cases for this equation.

The first one, if $n \equiv 0 \pmod 4$, then $x_n \equiv x_0 \equiv -2 \pmod{19}$. This implies $z^2 \equiv 8 \pmod{19}$, but the Legendre symbol $\left(\dfrac{8}{19}\right) = -1$.

The second case, if $n \equiv 1 \pmod 4$, then $x_n \equiv x_1 \equiv 4 \pmod{19}$. This implies $z^2 \equiv 288 \equiv 3 \pmod{19}$, but $\left(\dfrac{3}{19}\right) = -1$.

The third case, if $n \equiv 2 \pmod 4$, then $x_n \equiv x_2 \equiv 154 \equiv 2 \pmod{19}$. This implies $z^2 \equiv 4929 \equiv 8 \pmod{19}$ but we know from the first case $\left(\dfrac{8}{19}\right) = -1$.

The last case, if $n \equiv 3 \pmod 4$, then $x_n \equiv x_3 \equiv 5848 \equiv -4 \pmod{19}$. This implies $z^2 \equiv 288 \equiv 3 \pmod{19}$ and this is the same result with the second case.

Now if we consider the equation (8) ,we can calculate following table:

n	y_n	x_n
0	3	0
1	57	18
2	2163	684
3	82137	25974
4	3119043	986328

Table 4

From the *Lemma* 1, we get $x_{n+4} = 328776y_n + 1039681x_n$, so $x_{n+4} \equiv 53353x_n \pmod{328776}$ which implies $x_{n+4} \equiv x_n \pmod{19}$. Equation (5) becomes

$$z^2 \equiv 16 + 17x_n^2 \pmod{19}$$

We must check four different cases for this equation.

If $n \equiv 1 \pmod 4$, then $x_n \equiv x_1 \equiv 18 \pmod{19}$. This implies $z^2 \equiv 5524 \equiv 14 \pmod{19}$, but $\left(\dfrac{14}{19}\right) = -1$.

We know $x_{n+4} \equiv 53353x_n \pmod{328776}$,which implies $x_{n+4} \equiv -x_n \pmod{103}$. Equation (5) becomes

$$z^2 \equiv 16 + 17x_n^2 \pmod{103}$$

If $n \equiv 2 \pmod 4$ then $x_n \equiv x_2 \equiv 684 \equiv 66 \pmod{103}$. This implies $z^2 \equiv 74068 \equiv 11 \pmod{103}$, but $\left(\dfrac{11}{103}\right) = -1$.

If $n \equiv 3 \pmod 4$ then $x_n \equiv x_3 \equiv 25974 \equiv 18 \pmod{103}$. This implies $z^2 \equiv 5524 \equiv 65 \pmod{103}$, but $\left(\dfrac{65}{103}\right) = -1$.

If $n \equiv 0 \pmod 4$, $x_{n+4} = 328776y_n + 1039681x_n$ which implies $x_{n+4} = 328776y_n \pmod{1039681}$. If we use $x_{n+4} = 328776y_n \pmod{1039681}$ to the equation (4), we find $2y^2 \equiv 9 \pmod{1039681}$. Then we get $20x^2 \equiv -9 \pmod{1039681}$. The equation (5) implies that $z^2 \equiv 519924 \pmod{1039681}$, but Legendre symbol $\left(\dfrac{519924}{1039681}\right) = -1$, so (5) is impossible.

From all of the cases, the equation system can not be solved simultaneously, so this means the P_{-1} set $\{1, 10, 17\}$ can not be extended. This completes the proof. $\square$

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