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On a Class of Two Dimensional Twisted q-Tangent Numbers and Polynomials

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Abstract

In this paper we introduce two dimensional twisted q-tangent numbers and polynomials. We also give some properties, explicit formulas, several identities, a connection with two dimensional twisted q-tangent numbers and polynomials, and some integral formulas.

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1 Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, and tangent numbers(see [1, 2, 3, 4, 5, 7]). In this paper, we study some properties of a new type of two dimensional twisted q-tangent numbers and polynomials. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. For a real number(or complex number) x, q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q} \text{ if } q \neq 1, \quad [x]_q = x \text{ if } q = 1.$$

The q-binomial coefficients are defined for positive integer n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!},$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q, n = 1, 2, 3, \ldots$ and $[0]_q! = 1$, which is known as q-factorial(see [1]). The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{l}{2}} x^{n-l} y^l, n \in \mathbb{Z}_+.$$

For any $q \in \mathbb{C}$ with $|q| < 1$, the two form of q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \text{ and } E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

From this form we easily see that $e_q(z)E_q(-z) = 1$. The q-derivative operator of a any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0, \tag{1.1}$$

and $D_q f(0) = f'(0)$, provided that f is differentiable at 0. It happens clearly that $D_q x^n = [n]_q x^{n-1}$. Clearly, if the function $f(x)$ is differentiable on the point x, the q-derivative in (1.1) tends to the ordinary derivative in the classical analysis when q tends to 1. The q-tangent polynomials $\mathbf{T}_{n,q}(x)$ are defined by the generating function:

$$\left(\frac{2}{e_q(2t)+1}\right) e_q(xt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x) \frac{t^n}{[n]_q!} \quad (|2t| < \pi).$$

When $x = 0$, $\mathbf{T}_{n,q}(0) = \mathbf{T}_{n,q}$ are called the q-tangent numbers(see [4]). The two dimensional q-tangent polynomials $\mathbf{T}_{n,q}(x, y)$ in x, y are defined by means of the generating function:

$$\left(\frac{2}{e_q(2t)+1}\right) e_q(xt) e_q(yt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q}(x, y) \frac{t^n}{[n]_q!} \quad (|2t| < \pi).$$

The definite q-integral is defined as

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b). \tag{1.2}$$

Identically, if the function $f(x)$ is Riemann integrable on the concerned intervals, the q-integral in (1.2) tends to the Riemann integrals of $f(x)$ on the corresponding intervals when q tends to 1(see [1, 2, 5]). In the following section, we introduce the two dimensional twisted q-tangent numbers and polynomials. After that we will investigate some their properties. Finally, we give some relationships both between these polynomials and q-derivative operator and between these polynomials and q-integral.

2 Twisted q-tangent numbers and polynomials

In this section, we introduce the two dimensional twisted q-tangent numbers and polynomials and provide some of their relevant properties.

Let r be a positive integer, and let ζ be rth root of 1. The twisted q-tangent polynomials $\mathbf{T}_{n,q,\zeta}(x)$ are defined by the generating function:

$$\left(\frac{2}{\zeta e_q(2t)+1}\right) e_q(xt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x)\frac{t^n}{[n]_q!}. \tag{2.1}$$

When $x=0$, $\mathbf{T}_{n,q,\zeta}(0)=\mathbf{T}_{n,q,\zeta}$ are called the twisted q-tangent numbers. Upon setting $q=1$ in (2.1), we have

$$\left(\frac{2}{\zeta e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\zeta}(x)\frac{t^n}{n!},$$

where $T_{n,\zeta}(x)$ are called familiar twisted tangent polynomials. Setting $\zeta=1$ in (2.1), we can obtain the corresponding definitions for the q-tangent numbers and polynomials, respectively(see [4]). Numerous properties of twisted tangent numbers and polynomials are known. More studies and results in this subject we may see references [3], [4], [5], [6], [7]. About extensions for the tangent numbers can be found in [3, 4, 5, 6, 7].

The two dimensional twisted q-tangent polynomials $\mathbf{T}_{n,q,\zeta}(x,y)$ in x,y are defined by means of the generating function:

$$\left(\frac{2}{\zeta e_q(2t)+1}\right) e_q(xt)e_q(yt) = \sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x,y)\frac{t^n}{[n]_q!}. \tag{2.2}$$

It is obvious that $\lim_{q\to 1}\mathbf{T}_{n,q,\zeta}(x,y)=T_{n,\zeta}(x+y)$ and $\mathbf{T}_{n,q,\zeta}(x,0)=\mathbf{T}_{n,q,\zeta}(x)$.

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x)\frac{t^n}{[n]_q!} &= \left(\frac{2}{\zeta e_q(2t)+1}\right) e_q(xt) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}y^l\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients on both sides of (2.3), we have the following theorem.

Theorem 2.1 *For $n \in \mathbb{Z}_+$, we have*

$$\mathbf{T}_{n,q,\zeta}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta} x^l.$$

By using Definition of q-derivative operator and Theorem 2.1, we have the following theorem.

Theorem 2.2 *For $n \in \mathbb{Z}_+$, we have*

$$D_q \mathbf{T}_{n,q,\zeta}(x) = [n]_q \mathbf{T}_{n-1,q,\zeta}(x)$$

By Theorem 2.2 and Definition of the definite q-integral, we have

$$[n]_q \int_0^1 D_q \mathbf{T}_{n-1,q,\zeta}(x) d_q x = \mathbf{T}_{n,q,\zeta}(1) - \mathbf{T}_{n,q,\zeta}(0). \tag{2.4}$$

Since $\mathbf{T}_{n,q,\zeta}(0) = \mathbf{T}_{n,q,\zeta}$, by (2.4), we have the following theorem.

Theorem 2.3 *For $n \in \mathbb{Z}_+$, we have*

$$\int_0^1 D_q \mathbf{T}_{n-1,q,\zeta}(x) d_q x = \frac{\mathbf{T}_{n,q,\zeta}(1) - \mathbf{T}_{n,q,\zeta}}{[n]_q}.$$

Using the following identity:

$$\frac{2\zeta}{\zeta e_q(2t)+1} e_q(xt) e_q(2t) + \frac{2}{\zeta e_q(2t)+1} e_q(xt) = 2 e_q(xt),$$

we have the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$\zeta \mathbf{T}_{n,q,\zeta}(x,2) + \mathbf{T}_{n,q,\zeta}(x) = 2x^n.$$

Substituting $x = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5 *For $n \in \mathbb{Z}_+$, we have*

$$\zeta \mathbf{T}_{n,q,\zeta}(2) = -\mathbf{T}_{n,q,\zeta}.$$

By (2.2) and the rule of Cauchy product, we get

$$\sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}(x)y^l\right)\frac{t^n}{[n]_q!}. \tag{2.5}$$

By comparing the coefficients on both sides of (2.5), we have the following theorem.

Theorem 2.6 *For $n \in \mathbb{Z}_+$, we have*

$$\mathbf{T}_{n,q,\zeta}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}(x)y^l.$$

Using the following identity:

$$\frac{2\zeta}{\zeta e_q(2t)+1}e_q(2t) + \frac{2}{\zeta e_q(2t)+1} = 2,$$

we have the following theorem.

Theorem 2.7 *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \zeta 2^{n-l}\mathbf{T}_{l,q,\zeta} + \mathbf{T}_{n,q,\zeta} = \begin{cases} 2, & \text{if } n=0, \\ 0, & \text{if } n\neq 0. \end{cases}$$

By using definition of q-derivative operator, we have the following theorem.

Theorem 2.8 *For $n \in \mathbb{Z}_+$, we have*

$$D_{q,y}\mathbf{T}_{n,q,\zeta}(x,y) = [n]_q \mathbf{T}_{n-1,q,\zeta}(x,y).$$

3 Some identities involving twisted q-tangent numbers and polynomials

In this section, we give some relationships both between these polynomials and q-derivative operator and between these polynomials and q-integral. By (2.1) and by using Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x)\frac{t^n}{[n]_q!} &= \left(\frac{2}{\zeta e_q(2t)+1}\right)e_q(xt) \\ &= \left(\frac{2}{\zeta e_q(2t)+e_q(2t)e_{q^{-1}}(-2t)}\right)e_q(xt) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l\zeta^{-1}\mathbf{T}_{l,q^{-1},\zeta^{-1}}(2)x^{n-l}\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{3.1}$$

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1 *For $n \in \mathbb{Z}_+$, we have*

$$\mathbf{T}_{n,q,\zeta}(x) = \zeta^{-1} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l \mathbf{T}_{l,q^{-1},\zeta^{-1}}(2) x^{n-l}.$$

By Definition of the definite q-integral and Theorem 2.1, we get

$$\begin{aligned} \int_0^1 \mathbf{T}_{n,q,\zeta}(x) d_q x &= \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta} x^l d_q x \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta} \frac{1}{[l+1]_q}. \end{aligned} \tag{3.2}$$

We also get

$$\begin{aligned} \int_0^1 \mathbf{T}_{n,q,\zeta}(x) d_q x &= \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l \zeta^{-1} \mathbf{T}_{l,q^{-1},\zeta^{-1}}(2) x^{n-l} d_q x \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l \zeta^{-1} \mathbf{T}_{l,q^{-1},\zeta^{-1}}(2) \frac{1}{[n-l+1]_q}. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we have the following theorem.

Theorem 3.2 *For $n \in \mathbb{Z}_+$, we have*

$$\zeta \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{\mathbf{T}_{n-l,q,\zeta}}{[l+1]_q} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l \frac{\mathbf{T}_{l,q^{-1},\zeta^{-1}}(2)}{[n-l+1]_q}$$

Using the following identity:

$$\frac{2}{\zeta e_q(2t)+1} e_q(xt) e_q(yt) = \frac{2}{\zeta e_q(2t)+1} e_q(xt) \frac{\zeta e_q(\frac{t}{m}) - 1}{\frac{t}{m}} \frac{\frac{t}{m}}{\zeta e_q(\frac{t}{m}) - 1} e_q(\frac{t}{m} my),$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x,y) \frac{t^n}{[n]_q!} \\ &= \frac{m}{t} \left(\sum_{n=0}^{\infty} \mathbf{T}_{n,q,\zeta}(x) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} \zeta m^{-n} \frac{t^n}{[n]_q!} - 1 \right) \left(\frac{\frac{t}{m}}{\zeta e_q(\frac{t}{m}) - 1} e_q(\frac{t}{m} my) \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\zeta \mathbf{T}_{k+1,q,\zeta}(x, m^{-1}) - \mathbf{T}_{k+1,q,\zeta}(x))\, m^{k-n+1}}{[k+1]_q} \mathbf{B}_{n-k,q,\zeta}(my) \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

Matching the coefficient of $\dfrac{t^n}{[n]_q!}$ of both sides gives the following theorem.

Theorem 3.3 *For $n \in \mathbb{Z}_+$, we have*

$$\mathbf{T}_{n,q,\zeta}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\left(\zeta \mathbf{T}_{k+1,q,\zeta}(x, m^{-1}) - \mathbf{T}_{k+1,q,\zeta}(x)\right) m^{k-n+1}}{[k+1]_q} \mathbf{B}_{n-k,q,\zeta}(my)$$
$$= 2^n \mathbf{E}_{n,q,\zeta}\left(\frac{x}{2}, \frac{y}{2}\right).$$

Here $\mathbf{B}_{n,q,\zeta}(x,y)$ and $\mathbf{E}_{n,q,\zeta}(x,y)$ denote the twisted q-Bernoulli and twisted q-Euler polynomials in x, y which are defined by

$$\mathbf{B}_{n,q,\zeta}(x,y) = \frac{t}{\zeta e_q(t) - 1} e_q(xt) e_q(yt) \text{ and } \mathbf{E}_{n,q,\zeta}(x,y) = \frac{2}{\zeta e_q(t) + 1} e_q(xt) e_q(yt).$$

By Definition (2.1) and by using the following identity:

$$\frac{t}{\zeta e_q(t) - 1} e_q(xt) e_q(yt) = \frac{2}{\zeta e_q\left(2\frac{t}{m}\right) + 1} e_q\left(\frac{t}{m} my\right) \frac{\zeta e_q\left(2\frac{t}{m}\right) + 1}{2} \frac{t}{\zeta e_q(t) - 1} e_q(xt),$$

we get

$$\sum_{n=0}^{\infty} \mathbf{B}_{n,q,\zeta}(x,y) \frac{t^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathbf{B}_{k,q,\zeta}(x) \sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix}_q \zeta \mathbf{T}_{l,q,\zeta}(my) 2^{n-k-l-1} m^{k-n} \right) \frac{t^n}{[n]_q!}$$
$$+ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{-1} m^{k-n} \mathbf{B}_{k,q,\zeta}(x) \mathbf{T}_{n-k,q,\zeta}(my) \right) \frac{t^n}{[n]_q!}.$$

By comparing coefficients of $\frac{t^n}{[n]_q!}$ in the above equation, we arrive at the following theorem.

Theorem 3.4 *For $n \in \mathbb{Z}_+$, we have*

$$\mathbf{B}_{n,q,\zeta}(x,y)$$
$$= \frac{1}{2m^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathbf{B}_{k,q,\zeta}(x) \left[\sum_{l=0}^{n-k} \begin{bmatrix} n-k \\ l \end{bmatrix}_q \zeta 2^{n-k-l} m^k \mathbf{T}_{l,q,\zeta}(my) + m^k \mathbf{T}_{n-k,q,\zeta}(my) \right].$$

By Definition of the definite q-integral and Theorem 2.6, get

$$\int_0^1 y^n \mathbf{T}_{n,q,\zeta}(x,y) d_q y = \int_0^1 \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}(x) y^{l+n} d_q y$$
$$= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}(x) \frac{1}{[n+l+1]_q}. \tag{3.4}$$

By (1.2), we see that

$$\int_0^1 y^n \mathbf{T}_{n,q,\zeta}(x,y)d_qy = y^{n+1}\frac{\mathbf{T}_{n,q,\zeta}(x,y)}{[n+1]_q}\bigg|_0^1 - \int_0^1 [n]_q q^{n+1}y^{n+1}\frac{\mathbf{T}_{n-1,q,\zeta}(x,y)}{[n+1]_q}d_qy$$
$$= \frac{\mathbf{T}_{n,q,\zeta}(x,1)}{[n+1]_q} - \frac{q^{n+1}[n]_q\mathbf{T}_{n-1,q,\zeta}(x,1)}{[n+1]_q[n+2]_q}$$
$$+ (-1)^2\frac{q^{n+1}q^{n+2}[n]_q}{[n+1]_q}\frac{[n-1]_q}{[n+2]_q}\int_0^1 y^{n+2}\mathbf{T}_{n-2,q,\zeta}(x,y)d_qy.$$

Continuing this process, we obtain

$$\begin{aligned}\int_0^1 y^n\mathbf{T}_{n,q,\zeta}(x,y)d_qy &= \frac{\mathbf{T}_{n,q,\zeta}(x,1)}{[n+1]_q}\\ &+\sum_{m=1}^{n-1}\frac{q^{n+1}\cdots q^{n+m}[n]_q[n-1]_q\cdots[n-m+1]_q(-1)^m}{[n+1]_q[n+2]_q\cdots[n+m+1]_q}\mathbf{T}_{n-m,q,\zeta}(x,1)\\ &+(-1)^n\frac{q^{n+1}\cdots q^{2m}[n]_q!}{[n+1]_q[n+2]_q\cdots[2n]_q}\int_0^1 y^{2n}\mathbf{T}_{0,q,\zeta}(x,y)d_qy\end{aligned} \tag{3.5}$$

Hence, by (3.4) and (3.5), we have the following theorem.

Theorem 3.5 *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}\sum_{l=0}^{n}\begin{bmatrix} n \\ l\end{bmatrix}_q \mathbf{T}_{n-l,q,\zeta}(x)\frac{1}{[n+l+1]_q} &= \frac{\mathbf{T}_{n,q,\zeta}(x,1)}{[n+1]_q}\\ &+\sum_{m=1}^{n-1}(-1)^m\frac{q^{n+1}\cdots q^{n+m}[n]_q[n-1]_q\cdots[n-m+1]_q}{[n+1]_q[n+2]_q\cdots[n+m+1]_q}\mathbf{T}_{n-m,q,\zeta}(x,1)\\ &+(-1)^n\frac{q^{n+1}\cdots q^{2m}[n]_q!}{[n+1]_q[n+2]_q\cdots[2n]_q[2n+1]_q}.\end{aligned}$$

References

[1] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Vol. 71, Cambridge Press, Cambridge, UK, 1999. https://doi.org/10.1017/cbo9781107325937

[2] Y.H. Kim, H.Y. Jung, C.S. Ryoo, On the generalized Euler polynomials of the second kind, *J. Appl. Math. & Informatics*, **31** (2013), 623-630. https://doi.org/10.14317/jami.2013.623

[3] C.S. Ryoo, A note on the tangent numbers and polynomials, *Adv. Studies Theor. Phys.*, **7** (2013), no. 9, 447-454. https://doi.org/10.12988/astp.2013.13042

[4] C.S. Ryoo, Some properties of two dimensional q-tangent numbers and polynomials, *Global Journal of Pure and Applied Mathematics*, **12** (2016), 2999-3007.

[5] C.S. Ryoo, Differential equations associated with tangent numbers, *J. Appl. Math. & Informatics*, **34** (2016), 487-494. https://doi.org/10.14317/jami.2016.487

[6] C.S. Ryoo, A numerical investigation on the zeros of the tangent polynomials, *J. Appl. Math. & Informatics*, **32** (2014), 315-322. https://doi.org/10.14317/jami.2014.315

[7] H. Shin, J. Zeng, The q-tangent and q-secant numbers via continued fractions, *European J. Combin.*, **31** (2010), 1689-1705. https://doi.org/10.1016/j.ejc.2010.04.003

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