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Approximate Solutions for a Certain Class of Fractional Optimal Control Problems Using Laguerre Collocation Method

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Abstract

In this paper, an approximate formula of the fractional derivatives (Caputo sense) is derived. The proposed formula is based on the generalized Laguerre polynomials. Special attention is given to study the convergence analysis of the presented formula. The spectral Laguerre collocation method is presented for solving a class of fractional optimal control problems (FOCPs). The properties of Laguerre polynomials approximation and Rayleigh-Ritz method are used to reduce FOCPs to solve a system of algebraic equations which solved using Newton iteration method. Numerical results are provided to confirm the theoretical results and the efficiency of the proposed method.

Mathematics Subject Classification: 65N20, 41A30

Keywords: Fractional optimal control problems, Pseudo-spectral method, Laguerre polynomials, Rayleigh-Ritz method, Convergence analysis

1 Introduction

In this paper, we focus on optimal control problems with the quadratic performance index and the dynamic system with the Caputo fractional derivative. We implement the proposed algorithm for solving the following FOCPs:

$$\text{minimum } J = \frac{1}{2}\int_0^1 [p(t)x^2(t) + q(t)u^2(t)]dt, \tag{1}$$

subject to the system dynamics

$$D^\nu x(t) = a(t)x(t) + b(t)u(t), \tag{2}$$

with the initial condition $x(0) = x_0$, where $\nu > 0$ refers to the order of the Caputo fractional derivatives, $p(t) \geq 0,\ q(t) > 0,\ a(t) \neq 0$ and $b(t) \neq 0$ are given functions. Many authors studied these problems with different numerical methods. For more details about these problems, see ([5], [6], [8], [11])

2 Preliminaries and Notations

In this section, we present some necessary definitions & mathematical preliminaries of fractional calculus theory required for our subsequent development.

Definition 2.1 *The Caputo fractional derivative operator D^ν of order ν is defined in the following form*

$$D^\nu f(x) = \frac{1}{\Gamma(m-\nu)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}}dt, \quad m-1 < \nu \leq m,\ m \in \boldsymbol{N},\ x > 0.$$

For the Caputo's derivative we have $D^\nu C = 0$, C is a constant, and

$$D^\nu x^n = \begin{cases} 0, & \text{for} \quad n \in \boldsymbol{N}_0 \text{ and } n < \lceil\nu\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)}x^{n-\nu}, & \text{for} \quad n \in \boldsymbol{N}_0 \text{ and } n \geq \lceil\nu\rceil. \end{cases} \tag{3}$$

We use the ceiling function $\lceil\nu\rceil$ to denote the smallest integer greater than or equal to ν, and $\boldsymbol{N}_0 = \{0, 1, 2, ...\}$. For more details on fractional derivatives definitions and its properties see ([4], [12]).

The generalized Laguerre polynomials $[L_i^{(\alpha)}(x)]_{i=0}^{\infty}$, $\alpha > -1$ are defined on the unbounded interval $(0, \infty)$ and can be determined with the aid of the following recurrence formula ([3], [16])

$$(i+1)L_{i+1}^{(\alpha)}(x) + (x-2i-\alpha-1)L_i^{(\alpha)}(x) + (i+\alpha)L_{i-1}^{(\alpha)}(x) = 0, \qquad i = 1, 2, ..., \tag{4}$$

where, $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$.
The analytic form of these polynomials of degree n is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}x^k = \binom{n+\alpha}{n}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}, \tag{5}$$

$L_n^{(\alpha)}(0) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$. These polynomials are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = \frac{1}{\Gamma(1+\alpha)} x^{\alpha} e^{-x}$.

Any function $u(x)$ belongs to the space $L_w^2[0, \infty)$ of all square integrable functions on $[0, \infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x) = \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{6}$$

where the coefficients c_i are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} L_i^{(\alpha)}(x) u(x) dx, \qquad i = 0, 1, \dots . \tag{7}$$

Consider only the first $(m+1)$ terms of generalized Laguerre polynomials, so we can write

$$u_m(x) \cong \sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{8}$$

For more details on Laguerre polynomials, see ([9], [13]-[19]).

3 The Approximate Fractional Derivatives of $L_n^{(\alpha)}(x)$ and its Convergence Analysis

The main goal of this section is to introduce the following theorems to derive an approximate formula of the fractional derivatives of the generalized Laguerre polynomials and study the truncating error and its convergence analysis.

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 3.1 *[7] Let $u(x)$ be approximated by the generalized Laguerre polynomials as (8) and also suppose $\nu > 0$ then, its Caputo fractional derivative can be written in the following form*

$$D^{\nu}(u_m(x)) \cong \sum_{i=\lceil \nu \rceil}^{m} \sum_{k=\lceil \nu \rceil}^{i} c_i \, w_{i,k}^{(\nu)} \, x^{k-\nu}, \quad w_{i,k}^{(\nu)} = \frac{(-1)^k}{\Gamma(k+1-\nu)} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix}. \tag{9}$$

Theorem 3.2 *[7] The Caputo fractional derivative of order ν for the generalized Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form*

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=\lceil \nu \rceil}^{i} \sum_{j=0}^{k-\lceil \nu \rceil} \Omega_{ijk} \, L_j^{(\alpha)}(x), \qquad i = \lceil \nu \rceil, \lceil \nu \rceil + 1, \dots, m, \tag{10}$$

where $$\Omega_{i\,j\,k} = \sum_{r=0}^{j} \frac{(-1)^{r+k}\,(\alpha+i)!(j)!\,(k+\alpha-\nu+r)!}{(k-\nu)!(i-k)!\,(\alpha+k)!\,r!(j-r)!\,(\alpha+r)!}\,.$$

Remark 1:
We can deduce that the error in approximating $D^{\nu}u(x)$ by $D^{\nu}u_m(x)$ is bounded by [7]

$$|E_T(m)| \leq \sum_{i=m+1}^{\infty} c_i \Pi_{\nu}(i,j)\ \frac{(\alpha+1)_j}{j!}\, e^{x/2},\ \ \alpha \geq 0,\ \ \ j=0,1,2,\ldots, \tag{11}$$

$$|E_T(m)| \leq \sum_{i=m+1}^{\infty} c_i \Pi_{\nu}(i,j)\ \Big(2-\frac{(\alpha+1)_j}{j!}\Big)\, e^{x/2},\ \ -1<\alpha\leq 0,\ \ x\geq 0, \tag{12}$$

where, $\Pi_{\nu}(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \Omega_{i\,j\,k}$ and $|E_T(m)| = |D^{\nu}u(x) - D^{\nu}u_m(x)|$.

4 Procedure Solution using Laguerre Collocation Method and Numerical Results

In this section, we demonstrate the capability of the introduced approach with using the presented approximate formula of fractional derivative (9) to solve numerically the proposed problem of FOCPs defined in (1)-(2). To achieve this aim, we solve two widely used examples from the literature.

Problem 1 (Linear time-invariant problem)

Consider the following linear time invariant problem, which described by the following fractional optimal control problem ([1], [2])

$$\text{minimum } J = \frac{1}{2}\int_0^1 [x^2(t)+u^2(t)]dt, \tag{13}$$

subject to the system dynamics

$$D^{\nu}x(t) = -x(t)+u(t), \quad 0<\nu\leq 1, \quad x(0)=1. \tag{14}$$

Our aim is to find the control $u(t)$ which minimizes the quadratic performance index J.
The procedure of the presented algorithm is given by the following steps:

1. Substitute by Eq.(14) into Eq.(13) gives

$$J = \frac{1}{2}\int_0^1 [x^2(t)+(D^{\nu}x(t)+x(t))^2]dt. \tag{15}$$

2. Approximate the function $x(t)$ using Laguerre polynomials expansion with $m = 3$ as follows:

$$x_m(t) = \sum_{i=0}^{3} c_i L_i^{(\alpha)}(t), \tag{16}$$

and its Caputo fractional derivative $D^\nu x(t)$ using the proposed approximated formula (9), then Eq.(15) transformed to the following form

$$J = \frac{1}{2}\int_0^1 [\left(\sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2 + \left(\sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,k}^{(\nu)} t^{k-\nu} + \sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2]dt. \tag{17}$$

3. The integral term in Eq.(17) can be found using composite trapezoidal integration technique as

$$\int_0^1 \left[\left(\sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2 + \left(\sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,k}^{(\nu)} t^{k-\nu} + \sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2\right] dt$$

$$\cong \frac{h}{2}(\Omega(t_0) + \Omega(t_N) + 2\sum_{k=1}^{N-1}\Omega(t_k)), \tag{18}$$

where

$$\Omega(t) = \left(\sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2 + \left(\sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,k}^{(\nu)} t^{k-\nu} + \sum_{i=0}^{3} c_i L_i^{(\alpha)}(t)\right)^2,$$

$h = \frac{1}{N}$, for an arbitrary integer N, $t_i = ih, \ \ i = 0, 1, ..., N$.
So, we can write Eq.(18) in the following approximated form

$$J(c_0, c_1, c_2, c_3) = \frac{h}{4}[\Omega(t_0) + \Omega(t_N) + 2\sum_{k=1}^{N-1}\Omega(t_k)]. \tag{19}$$

4. The extremal values of functionals of the general form (19), according to Rayleigh-Ritz method gives

$$\frac{\partial J}{\partial c_0} = 0, \quad \frac{\partial J}{\partial c_1} = 0, \quad \frac{\partial J}{\partial c_2} = 0, \quad \frac{\partial J}{\partial c_3} = 0,$$

so, after using the boundary conditions, we obtain a system of nonlinear algebraic equations.

5. Solve the resulting non-linear system of algebraic equations using Newton iteration method to obtain c_0, c_1, c_2, c_3, then the function $x(t)$ which extremes FOCPs (13) has the form (16). Therefore, the control $u(t)$ will obtain as follows

$$u(t) = (D^\nu x(t) + x(t))^2. \tag{20}$$

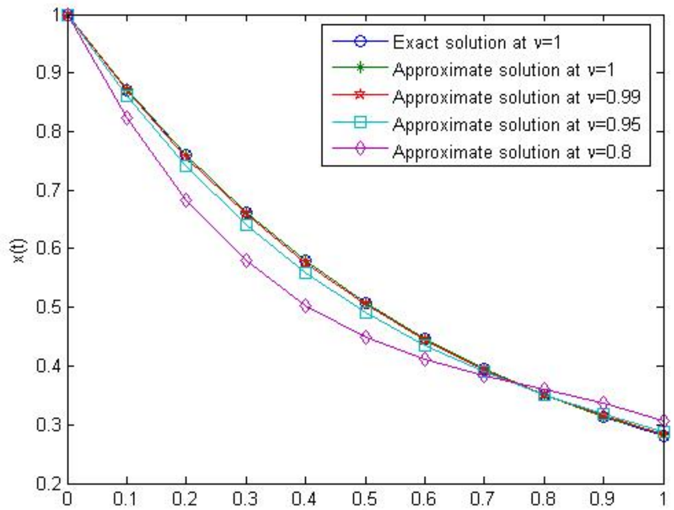

Figure 1. The behavior of $x(t)$ for problem 1 at $m = 3$ with different values of ν.

The behavior of the numerical solutions of this problem with different values of m and ν are given in figures 1-3. Where in figures 1-2, the numerical solutions $x(t)$ and $u(t)$, respectively at $m = 3$ for different values of ν and the exact solution at $\nu = 1$ are plotted. In figure 3, the numerical solutions $x(t)$ at $\nu = 0.8$ for different values of $m(m = 3, 4, 5)$ are plotted.

The solution obtained using the presented method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (16). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique and the advantage this method from the other methods [10].

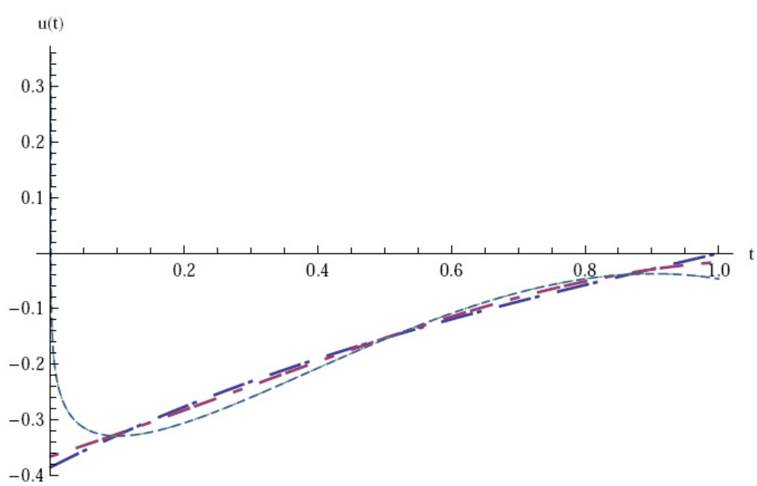

Figure 2. The behavior of $u(t)$ for problem 1 at $m = 3$ with different values of ν.

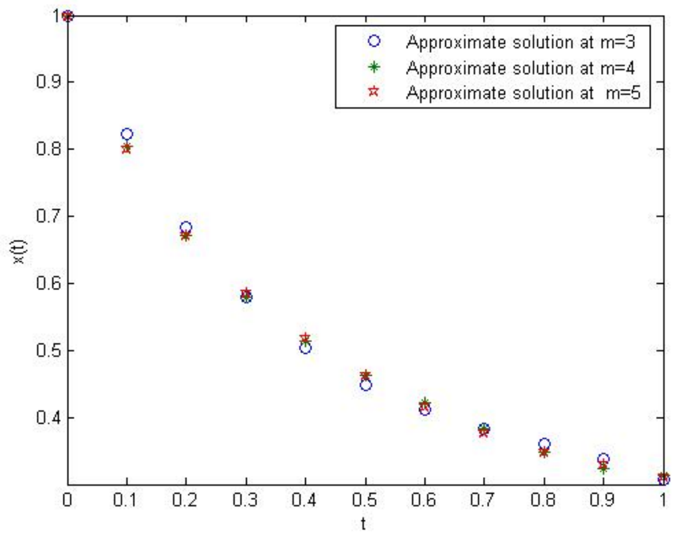

Figure 3. The behavior of $x(t)$ for problem 1 at $\nu = 0.8$ with $m = 3, 4\,\text{and}\,m = 5$.

Problem 2 (Linear time-variant problem)

In this example, we consider the linear time variant FOCP [2]

$$\text{minimum } J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt, \tag{21}$$

subject to the system dynamics

$$D^\nu x(t) = t\,x(t) + u(t), \quad x(0) = 1. \tag{22}$$

Our aim is to find the control $u(t)$ which minimizes the quadratic performance index J. We will implement the proposed algorithm as described in the previous problem with $m = 3$.

The behavior of the numerical solutions of this problem with different values of m and ν are given in figures 4 and 5. Where in figure 4, the numerical solutions $x(t)$ at $m = 3$ for different values of ν and the exact solution at $\nu = 1$ are plotted. In figure 5, the numerical solutions $x(t)$ at $\nu = 0.8$ for different values of $m(m = 3, 4, 5)$ are plotted.

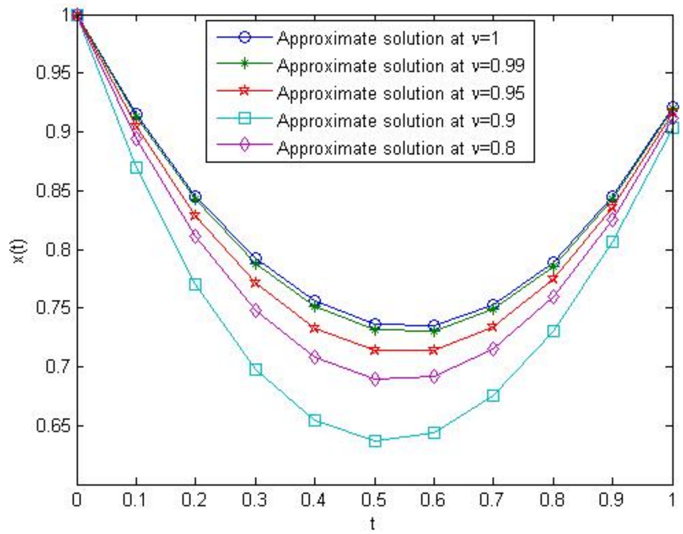

Figure 4. The behavior of $x(t)$ for problem 2 at $m = 3$ with different values of ν.

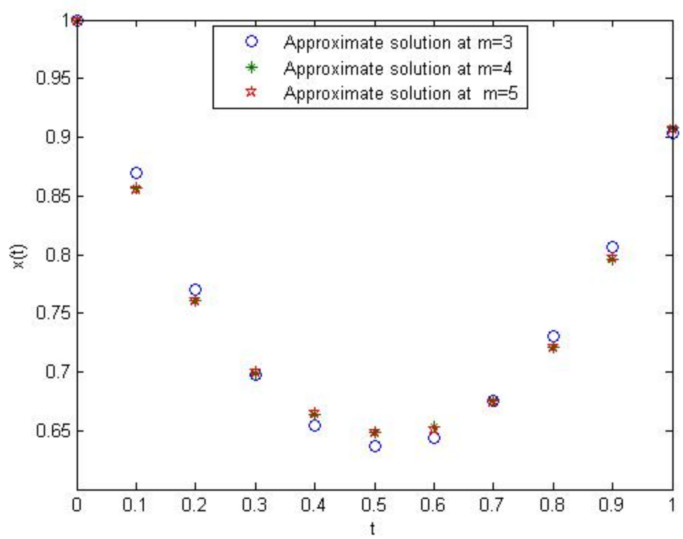

Figure 5. The behavior of $x(t)$ for problem 2 at $\nu = 0.8$ with $m = 3, 4$ and $m = 5$.

5 Conclusion and Remarks

In this article, we introduced an accurate numerical technique for solving a certain class of fractional optimal control problems. We have introduced an approximate formula for the Caputo fractional derivative of the generalized Laguerre polynomials in terms of generalized Laguerre polynomials themselves. In the proposed method, the properties of the Laguerre polynomials and Rayleigh-Ritz method are used to reduce the FOCP to solve a system of algebraic equations. The error upper bound of the proposed approximate formula is stated and discussed. The results show that the algorithm converges as the number of m terms is increased. The solution is expressed as a truncated Laguerre series and so it can be easily evaluated for arbitrary values of time using any computer program without any computational effort. From illustrative examples, we can conclude that this approach can obtain very accurate and satisfactory results. For all examples, the solution for the integer order case of the problem is also obtained for the purpose of comparison. Finally, from our numerical results using the proposed method, we can see that, the solutions are in excellent agreement with the exact solution and better than the numerical results obtained in Agrawal and Lotfi approaches. Also, from the proposed examples, we can conclude that the presented method is difficult to extend on fractional optimal control problems with non-quadratic objective, and also problems with non-linear right hand side of the differential equations.

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