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Reorder Quantities for (Q, R) Inventory Models

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Abstract

The reorder level R and the reorder quantity Q are the parameters to be decided in a continuous review inventory policy. Their optimal values can be approached through iterative methods, but these are tedious and inconvenient for control routines. A frequent practice is to set Q as the economic reorder quantity and compute R accordingly. Yet this practice may introduce a substantial cost penalty. Existing literature demonstrated attempts to rearrange conventional theoretical expressions to facilitate the use of numerical approximations to help find an optimal solution in a continuous review inventory policy, however, the cost function derived and used in the rearrangement were faulty. This paper re-visits the formulation of the cost function by stating explicitly necessary assumptions, and obtained the correct cost function. Expressions for the optimising parameters R & Q were re-obtained based on the correct cost function. The new optimal expressions for R & Q and the cost function were implemented. The result of this work showed that, the reviewed model gives lower inventory cost with higher optimal order quantity than the model under review, and hence the reviewed model performs better when compared to the model under review.

Mathematics Subject Classification: Primary 92D40, 92D25; Secondary: 34D20

Keywords: Inventory, Continuous review, stochastic, Re-order quantity, Re-order level, Wilson's EOQ, backlogging, Demand, probabilistic

Introduction

Controlling stock (inventory) is a process and a method of total stock management. A lot of research efforts have been devoted to stock (inventory) control over the past few decades. Previous researches assume that the lead time is an incontrollable variable, unmet demand is always backordered, and so on. However, in reality these assumptions may not be tenable; this leads to the need to reconsider the inventory control problem. Some studies relevant to continuous review inventory models with stochastic demand were carried out by [9], [10], [8] and [3]. Moreover, many companies have recognized the importance of response time as a competitive weapon; and have used it as a means of differentiating themselves in the marketplace. In some previous studies, lead time is viewed as a prescribed constant or a random variable, which therefore is not subject to control. In many practical situations, lead time can be reduced at an added crashing cost; in other words, [12], [11].

In stochastic inventory control models under continuous review technique, the reorder level R and the order quantity Q are the parameters to be determined optimally, their optimal values have to be obtained iteratively in most cases, but this is tedious and inconvenient for control routines. In some cases, a practice of setting Q as the economic re-order quantity and computing R accordingly is employed, this practice may have a substantial cost penalty because Q was not obtained through optimal procedures, the problem is explicit expressions for Q and R cannot be obtained, [1]. [1] studied (Q, R) inventory model and proposed a non-iterative method of determining the optimal parameters of the model, their model uses numerical approximation rather than iteration. They obtained their model's cost function as:

$C_{Q,R} = \frac{(S+sz)}{Q} D + hc\left(\frac{Q}{2} + R - d\right)$, at optimal points, they obtained:

$Q = \sqrt{\frac{2D(S+sz)}{hc}} \quad and \quad P_r = \frac{hcQ}{sD}$

Where: Q – – – – – – – – – – – – – – Reorder quantity

R – – – – – – – – – – – – – – Reorder level

$P_r = \int_R^{\infty} f(x)dx$ – – – – – Probability of stock out

h – – – – – – – – – – – – – – Holding Cost

D – – – – – – – – – – – – – – Annual average demand

$Z = \int_R^{\infty} (x - R) f(x)dx$ – – – Expected shortage per cycle

S – – – – – – – – – – – – – – Cost per order

s – – – – – – – – – – – – – – Shortage cost per physical unit short

$P_p = \frac{Z}{Q}$ – – – – – – – – – – – – Expected shortage rate

x – – – – – – – – – – – – – – Lead-time demand

$f(x)$ – – – – – – – – – – – – – – Density distribution function

d – – – – – – – – – – – – – – Lead time mean demand

σ – – – – – – – – – – – – – – Lead-time standard deviation

$g = \left(\frac{d}{\sigma}\right)^2 - - - - - - - - - -$ The modulus of the gamma distribution

$Q_w = \sqrt{\frac{2SD}{hc}} - - - - - - - - -$ Economic order quantity (Wilson model)

After mathematical rearrangement, [1] obtained a standardised expression for Q as;

$$\frac{Q}{d} = \frac{Z}{dP_r} + \sqrt{\left(\frac{Z}{dP_r}\right)^2 + \left(\frac{Q_w}{d}\right)^2} \quad (1)$$

This is the expression used in their numerical approximations.

[5] attempted to apply the fuzzy set concepts to deal with uncertain backorders and lost sales. The purpose of their paper was to modify Moon and Choi S (1998)'s continuous review inventory model with variable lead time and partial backorders by fuzzifying the backorder rate (or equivalently, fuzzifying the lost sales rate). They first considered the case where the lost sales rate is treated as the triangular fuzzy number. Then, through the statistical method for establishing the interval estimation of the lost sales rate, they constructed a new fuzzy number, namely statistic-fuzzy number. For each fuzzy case, they investigated a computing schema for the modified continuous review inventory model and develop an algorithm to find the optimal inventory strategy.

[2] in his work developed a mathematical model for continuous-review reorder quantity (r, Q) inventory system. In his paper, he took into account the continuous-review reorder point-lot size under stochastic demand, with the backorders-lost sales mixture. To minimize the expected total cost per time unit, they then propose an exact algorithm and a heuristic procedure. The heuristic method exploits an approximated expression of the total cost function achieved by means of an ad hoc first-order Taylor polynomial. They carried out numerical experiments with a twofold objective. On the one hand they examine the efficiency of the approximated solution procedure, while on the other hand they investigated the performance of the maximum entropy principle in approximating the true lead-time demand distribution.

[6] discussed the effects of ramp-type of demand on an EOQ model for non-instantaneous Weibull deteriorating items under time-dependent partial backlogging. They considered a retailer who purchases and sells a single product over an infinite time horizon who faces a ramp-type of demand. They allowed shortages with partially backlogging with time to the deterioration of the item distributed as Weibull. They considered the Total cost and the optimum number of replenishments as decision variables. They obtained the optimal replenishment policies that minimise the total cost and demonstrated the performance of the model with a numerical samples are explained in the following sections.

[7] considered the management of safety stock in a coordinated single-vendor single-buyer supply chain under continuous review and Gaussian lead-time demand. They assumed the lead time to be controllable, and shortages not allowed. They followed value criterion by considering both inflation and time value of money. The aim was to present a novel approach to optimizing the safety

stock in such system In their paper, they took a different perspective by putting the order quantity and the safety factor in functional dependence through the adoption of a specific parameter. More precisely, they expressed the service level as a function of the number of admissible stockouts per time unit and the order quantity. This allows optimizing the safety stock taking into account the constraint on the number of admissible stockouts per time unit. They present both exact and approximated minimization algorithms. Numerical examples were finally shown to illustrate the effectiveness of the approximation algorithm, and to investigate the sensitivity of the model with respect to variations in some fundamental parameters.

In this paper, we revisit the paper by [1] in the face of the reviewed literatures above to correct the cost components to obtain the correct formulations. [1] gave their Model's cost function as;

$$C_{Q,R} = \frac{(S + sz)}{Q} D + hc\left(\frac{Q}{2} + R - d\right)$$

where; $\frac{SD}{Q}$ $- - - - - - - - - - - - - - -$ Total ordering cost component

$\frac{szD}{Q}$ $- - - - - - - - - - - - - - -$ Total shortage cost component

$ch\left(\left(\frac{Q}{2}\right) + r - d\right) - - - - - - - -$ Total holding cost component

The third (Total holding cost) component needs to be reformulated for the following reasons.

i. Multiplying $h\big((Q/2) + R - d\big)$ by 'c' is not correct, this is because 'c' is a unit item cost and we don't use unit item cost of stock ordered in determining inventory holding cost, rather we use the unit item holding cost like storage cost, heating cost, cooling cost e.t.c. incurred in holding the inventory before depletion.

ii. Using $Q/2$ in the holding cost expression has no significance other than just to make it possible to use Wilson's EOQ model to approximate and simplify computations, you can use $Q/2$ only when demand rate is known to be constant, i.e. stock is withdrawn from the inventory at a constant rate, in this case the average number of unit in inventory over time T is $Q/2$.

New Model Formulation

Assumptions

1) Shortage is allowed with backlogging
2) Demand is probabilistic
3) Minimization of the average cost is the objective
4) Planning horizon is infinite
5) Replenishment is instantaneous
6) Cost of replenishment is independent of the quantity ordered
7) Lead time is finite

Notations

We maintain the same notations used by [1] as follows

Q – – – – – – – – – – – – – – – – Reorder quantity

R – – – – – – – – – – – – – – – – Reorder level

$P_r = \int_R^{\infty} f(x)\,dx$ – – – – – – – – – Probability of stock out

h – – – – – – – – – – – – – – – – Holding Cost

c – – – – – – – – – – – – – – – – Item unit Cost

D – – – – – – – – – – – – – – – – Annual average demand

$Z = \int_R^{\infty}(x - R)f(x)dx$ – – – – – Expected shortage per cycle

S – – – – – – – – – – – – – – – – Cost per order

s – – – – – – – – – – – – – – – – Shortage cost per physical unit short

$P_p = \frac{Z}{Q}$ – – – – – – – – – – – – – – Expected shortage rate

X – – – – – – – – – – – – – – – – Lead-time demand

$f(x)$ – – – – – – – – – – – – – – – – Density distribution function

d – – – – – – – – – – – – – – – – – Lead time mean demand

σ – – – – – – – – – – – – – – – – – Lead-time standard deviation

$g = (d/\sigma)^2$ – – – – – – – – – – – – The modulus of the gamma distribution

$Q_w = \sqrt{\frac{2SD}{hc}}$ – – – – – – – – – – – – – Economic order quantity (Wilson model)

Derivation of the new model

In deriving the mathematical model, we approach the problem by obtaining the individual cost components as follows;

$\frac{SD}{Q}$ – – – – – – – – – – – – – – – – Total ordering cost component

$\frac{szD}{Q}$ – – – – – – – – – – – – – – – Total shortage cost component

$h(Q + r - d)$ – – – – – – – – – – Total holding cost component

Therefore the new cost function is the sum of the above cost components as follows; $C_{Q,R} = \frac{(S+sz)}{Q}D + h(Q + R - d)$ (2)

The optimal (in this case minimum) value for $C_{Q,R}$ is obtained by setting to zero (0) its Q & R derivatives. Thus obtaining the new expressions for Q and P_r as follows;

$$Q = \sqrt{\frac{D(S+sz)}{h}} \quad (3)$$

and

$$P_r = \frac{hQ}{sD} \quad (4)$$

Standard formulation

Equation (3) & (4) can be rearranged to standardise computing procedure. From (3) we have $Q^2 = SD/h + szD/h = 2DSc/2hc + \frac{szD}{h} = \frac{(Q_w)^2 c}{2} + szD/h.$

Here Q_w is the Wilson's EOQ

$= \frac{(Q_w)^2 c}{2} + \frac{zQ}{P_r}$: Since $Ds/h = Q/P_r$ from equation (3)

Therefore $Q^2 - \frac{zQ}{P_r} - \frac{(Q_w)^2 c}{2} = 0$

By completing squares on$Q$ and solving for Q, we have

$$Q = \frac{Z}{2P_r} + \sqrt{\left(\frac{z}{2P_r}\right)^2 + (Q_w)^2 \frac{c}{2}} \tag{5}$$

Dividing through by d to get the new standardised expression for Q, we have

$$\frac{Q}{d} = \frac{Z}{2dP_r} + \sqrt{\left(\frac{z}{2dP_r}\right)^2 + \left(\frac{Q_w}{d}\right)^2 \frac{c}{2}} \tag{6}$$

Equation (6) is the equivalence of equation (2) in the model by [1]. Now, we apply the new model to the same Gamma distributed demands as follows;
If lead time demand follows a –ve exponential distribution, i.e. $d = \sigma$, then the modulus of the gamma distribution $g = \left(\frac{d}{\sigma}\right)^2 = 1$, this implies that $\frac{z}{P_r} = d$. Hence we have

$$\frac{Q}{d} = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{Q_w}{d}\right)^2 \frac{c}{2}} \tag{7}$$

which is the equivalence of $\frac{Q}{d} = 1 + \sqrt{1 + \left(\frac{Q_w}{d}\right)^2}$

In [1] this expression gives the optimum Q directly, whatever the value of Pr.
As in BC Vasconcelos and MP Marques (2000), the ratio z/dP_r for other distributions other than $-ve$ exponential, strongly depend on g, depends only slightly on P_r within the usual range adopted in practice, say $5\% < P_r < 35\%$. As in [1], the values for z/dP_r were computed through numerical approximations borrowed from [4].

After obtaining Q, the subsequent step is the computation of the reorder level R, we are going to adopt the same procedure as in [1] for determining Q.

Hypothetical Example:
We are going to use the same example as in [1] to enable us asses the performance of the model.

Given a continuous review inventory system and let $c = \$15, S = \$30, h = 20\%$ $s = \$12, D = 1000\ units\ p.a., d = 100,$
$\sigma = 120$. We are to compute Q and R using the new model

Solution
The computed values of $Q_w, g, \frac{z}{dP_r}, \frac{z}{2dP_r},$ are as follows

$$Q_w = 158,\ \ g = 0.7,\ \ \frac{z}{dP_r} = 1.30,\ \ \frac{z}{2dP_r} = 0.65$$

Therefore, $\frac{Q}{d} = 0.65 + \sqrt{7.5\left(\frac{158}{100}\right)^2 + (0.65)^2} = 0.65 + \sqrt{18.723 + 0.4225}$

$$Q = (0.65 + 4.38) * 100 = 503$$

Now computing P_r from $P_r = hQ/sD$, we have

$$P_r = 0.2 * \frac{503}{12*1000} = 0.08$$

Now, adapting Johnston approximations as follows:

We assume gamma distribution, $0.5 < g < 12, P_r < 0.5$

$$R/\sigma = A_3 + A_4 \ln(P_r) + A_5 (P_r)^2 + A_6 P_r \ln(P_r)$$

where $A_3 = 0.010179 - 0.0156841 * g^2 + 1.66011 * [\ln(g)]^2 + 0.145241 * g * \ln(g)$

$$A_4 = -0.998223 - 0.00231704 * g^2 + 0.357714 * \ln(g) - 0.106577 * [\ln(g)]^2 + 0.0201662 * g * \ln(g)$$

$$A_5 = -1.48338 - 0.000741918 * g^2 + \frac{1.46426}{g} - 0.206282 * \ln(g)$$

$$A_6 = 2.76031 - 2.72033 * g - 0.0544844 * g^2 + 3.13504 * \ln(g) + 1.04581 * g * \ln(g)$$

$$\therefore \quad \frac{R}{\sigma} = [0.0106179 - 0.0156841 * (0.49) + 1.66011 * \ln(0.7) - 0.365992 * \{\ln(0.7)\}^2 + 0.145241 * (0.7) * \ln(0.7)] + [-0.998223 - 0.00231704 * 0.49 + 0.357714 * \ln(0.7) - 0.106577 * \{\ln(0.7)\}^2 + 0.0201662 * (0.7) * \ln(0.7)] \ln(0.08) + \left[-1.48338 - 0.000741918 * 0.49 + \frac{1.46426}{0.7} - 0.206282 * \ln(0.7)\right] * (0.0064) + [2.76031 - 2.72033 * (0.7) - 0.0544844 * 0.49 + 3.13504 * \ln(0.7) + 1.04581 * (0.7) * \ln(0.7)] * 0.08 * \ln(0.08)$$

$$= -0.6720 + 2.8933 + 0.0034 + 0.1111 = 2.3358$$

Therefore $R = 2.3358 * 120 = 280.296 \approx 280$, $\frac{Z}{d} = 0.104 \Rightarrow Z = 10.4$

$$\therefore \quad C_{503,280} = 30 * \frac{1000}{508} + 12 * 10.4 * \frac{1000}{508} + 0.2(508 + 280 - 100)$$

$$= \$442.32$$

Discussion

1) **The Cost functions, Reorder quantity and Probability of stock out**

This paper revisited the inventory problem by [1] and obtained the correct expressions for $C_{Q,R}$, Q and P_r, respectively as follows;

$$C_{Q,R} = \frac{(S+sz)}{Q} D + hc(Q + R - d), \quad Q = \sqrt{\frac{D(S+sz)}{h}} \quad and \quad P_r = \frac{hQ}{sD}$$

2) **The standard formulated expressions**

This paper re-obtained the standardized expression, by [1], for Q based on the correct cost function as: $\frac{Q}{d} = \frac{Z}{2dP_r} + \sqrt{\left(\frac{Z}{2dP_r}\right)^2 + \left(\frac{Q_w}{d}\right)^2 \frac{c}{2}}$ instead of $\frac{Q}{d} = \frac{Z}{dP_r} + \sqrt{\left(\frac{Z}{dP_r}\right)^2 + \left(\frac{Q_w}{d}\right)^2}$

3) **Calculating Q_w and hence that Q, R and P_r**

In this paper, the value of Q_w was recomputed to obtained the correct value of Q_w as $Q_w = 158$ units instead of $Q_w = 141$ as in [1]. The values of P_r and R were also recomputed based on the new value $Q_w = 158$ units to obtain $P_r = 0.08$ and $R \approx 280$.

4) **The performance of the models**

i) The new model gives higher optimal order quantity of 508 units against 320 units from [1] model at the same reorder level.

ii) The optimal cost of the inventory with the new model is \$442.32 as against \$1,526 using the same example by [1].

iii) Clearly the new model gives lower inventory cost with higher optimal order quantity than the model by [1], which gives an inventory cost of $1,526 with an optimal order quantity of 320 units with the same example.

Conclusion

From the above discussion, we see that the result of this work demonstrated that, the reviewed model gives lower inventory cost with higher optimal order quantity than the model under review, and hence the reviewed model performs better when compared to the model under review.

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