Nonl. Analysis and Differential Equations, Vol. 1, 2013, no. 2, 93 - 100
HIKARI Ltd, www.m-hikari.com

Periodic Orbits of Weakly Exact Magnetic Flows

Osvaldo Osuna

Instituto de Física y Matemáticas, Universidad Michoacana
Edif. C-3, Cd. Universitaria
C.P. 58040. Morelia, Michoacán, México
osvaldo@ifm.umich.mx

Abstract

For a weakly exact magnetic flows with a bounded primitive on a closed Riemannian manifold, we prove the existence of periodic orbits in almost all energy levels below of the Mañé's critical value.

Mathematics Subject Classification: 37J45

Keywords: Euler-Lagrange flow, Mañé's critical value, magnetic flows, Palais-Smale condition

1 Introduction and results

Let (M, g) be a closed Riemannian manifold, denote by ω_0 the symplectic form on TM, obtained by pulling back the canonical symplectic form of T^*M via g. Let Ω be a closed 2-form on M and $\omega := \omega_0 + \pi^*\Omega$, where $\pi : TM \to M$ is the canonical projection. Consider $F : TM \to \mathbb{R}$, $F(x, v) = \frac{1}{2}g_x(v, v)$. The *magnetic flow* of (g, Ω) is the Hamiltonian flow of F with respect to ω.

We say that a magnetic flow is *weakly exact*, if the pull-back $\widetilde{\Omega}$ of Ω to the universal cover $\widetilde{M}$ is exact. Suppose that $\widetilde{\Omega}$ is exact with a smooth 1-form θ such that $\widetilde{\Omega} = d\theta$. We consider the Lagrangian on $\widetilde{M}$ given by $L(x, v) = \frac{1}{2}\widetilde{g}_x(v, v) + \theta_x(v)$, where $(\widetilde{M}, \widetilde{g})$ is a Riemannian universal covering of (M, g), thus the flow of L coincide with the lift of the magnetic flow (g, Ω) therefore is complete.

Recall that on a boundaryless, complete Riemannian manifold N an *autonomous Lagrangian* is a smooth function, $L : TN \to \mathbb{R}$ such that L is convex and superlinear when restricted to any fiber (see [2]), i.e., L is superlinear if for all $A \in \mathbb{R}$ there is $B \in \mathbb{R}$ such that $L(x, v) \geq A|v| - B$ for all $(x, v) \in TN$. Note if a weakly exact magnetic flow has a bounded primitive θ, then our Lagrangian L satisfies the above conditions (see [2] for details).

The energy $E : TN \to \mathbb{R}$ of L is given by $E(x,v) := \frac{\partial L}{\partial v}(x,v)v - L(x,v)$, E is a first integral for the flow φ_t of L. The *L-action* of an absolutely continuous curve $\gamma : [a,b] \to \widetilde{M}$ is defined by $S_L(\gamma) := \int_a^b L(\gamma(t), \dot\gamma(t))dt$. The *Mañé's critical value* is given by $c(L) := \inf\{ k \in \mathbb{R} \mid S_{L+k}(\gamma) \geq 0 \}$, where γ is any absolutely continuous closed curve defined on any closed interval $[a,b]$. It is well-known that $c(L) \geq e_0 := \min\{ k \in \mathbb{R} \mid \pi(E^{-1}(k)) = \widetilde{M}\}$. For weakly exact magnetic flow $e_0 = 0$.

Our interest is to study the existence problem for periodic orbits of weakly exact magnetic flows, we study the case of lower energy levels where very little is known, recall that, the exact case, was treated by G. Contreras [1], for some related results in small energy levels see [4], [3]. The last results use symplectic techniques which do not give a precise estimates of the energy of the periodic orbits. Being able to express the weakly exact magnetic flow as a Lagrangian allows us to use all the variational techniques, thus we have

Theorem 1.1 *A magnetic flow given by a magnetic field whose pullback to the universal cover has bounded primitive admits a contractible periodic orbit with energy k and positive $(L+k)$-action for almost $k \subset (e_0, c(L))$ with respect to the Lebesgue measure, where L is the corresponding Lagrangian on the universal cover.*

Remarks: The methods for to prove ours results are an adaptation from those used in [1]. I thank to G. Contreras for his suggestion to work this case.

2 Variational Setting

Let $\mathcal{H}^1(\widetilde{M}) := \{x : [0,1] \to \widetilde{M} \mid$ a. c. curve with $\int_0^1 |\dot x(s)|^2_{x(s)} ds < \infty \}$, it is a Hilbert manifold and its tangent space at the curve $x(s)$ is given by weakly differentiable vector fields along $x(s)$ with covariant derivative in $\mathcal{L}^2$. We consider in the Hilbert manifold $\mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+$ the Riemannian metric

$$\langle (\xi,\alpha),(\eta,\beta)\rangle_{(x,T)} := \alpha\beta + f^2(T)\langle \xi(0),\eta(0)\rangle + f(T)\int_0^1 \langle \frac{D}{ds}\xi(s), \frac{D}{ds}\eta(s)\rangle_{x(s)}\, ds,$$

where $\frac{D}{ds}$ is the covariant derivative along $x(s)$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth, positive function with $f(T) := T$ if $T \leq 1$ and equal to 1 if $T \geq 10$. Observe that this metric is locally equivalent to the metric obtained with $f = 1$. Let $\Lambda_{\widetilde{M}}$ be the set of closed curves in $\mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+$ it is a Hilbert submanifold of $\mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+$. Given $k \in \mathbb{R}$, we define the *action functional* $\widehat{\mathcal{S}}_k : \mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+ \to \mathbb{R}$ as

$$\widehat{\mathcal{S}}_k(x,T) := \int_0^1 [\ L(x(s), \frac{\dot x(s)}{T}) + k\ ]T\, ds = \int_0^1 [\ \frac{1}{2T}|\dot x(s)|^2 + \theta_{x(s)}(\dot x(s)) + kT\ ]\, ds.$$

where $(x,T) \in \mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+$, if $y(t) := x(\frac{t}{T}), t \in [0T]$ then $\widehat{\mathcal{S}}_k(x,T) = S_{L+k}(y)$. From a Smale theorem, we have that $\widehat{\mathcal{S}}_k$ is C^2 on $\mathcal{H}^1(\widetilde{M}) \times \mathbb{R}^+$ (see [5]) and the derivative of $\widehat{\mathcal{S}}_k$ is given by

$$d_{(x,T)}\widehat{\mathcal{S}}_k(\xi,\alpha) = \zeta \cdot L_v(y,\dot{y})|_0^T + \int_0^T \zeta \cdot [\frac{d}{dt}L_v - L_x]_{(y,\dot{y})} ds + \frac{\alpha}{T}\int_0^T [k - E(y,\dot{y})]ds,$$

where $y(t) := x(\frac{t}{T}), \zeta(t) := \xi(\frac{t}{T})$, for $0 \le t \le T$ and $E(x,v) = vL_v(x,v) - L(x,v) = \frac{1}{2}|v|_x^2$. Note in a critical point, the second term will be zero which implies that $y(t)$ is a solution of the Euler-Lagrange equation, therefore $y(t)$ has constant energy and the third term implies that this energy is equal to k.

Now, taking $(x_1,T) \in \Lambda_{\widetilde{M}} \times \mathbb{R}^+$ and h a deck transformation. We consider $x_2 := h \circ x_1$ then $\widehat{\mathcal{S}}_k(x_1,T) = \widehat{\mathcal{S}}_k(x_2,T)$. Indeed, let $p : \widetilde{M} \to M$ be the canonical projection then the conditions $\widetilde{\Omega} = p^*\Omega$ and $p = p \circ h$ imply that $\widetilde{\Omega} = h^*\widetilde{\Omega}$. Now, since $\widetilde{M}$ is simply connected the closed curves x_i, can be extended to a smooth map $\phi_i : \mathbb{D}^2 \to \widetilde{M}$ such that $\phi_i|_{\partial\mathbb{D}^2} = x_i$ respectively for $i = 1,2$. For any such extension ϕ_i we have $\int_{\mathbb{D}^2} \phi_i^*(\widetilde{\Omega}) = \int_{x_i} \theta$. In particular, if we take $\phi_2 := h \circ \phi_1$ then $\int_{x_2} \theta = \int_{x_1} \theta$. Finally, by taking a metric $\widetilde{g}$ on $\widetilde{M}$ such that $(\widetilde{M},\widetilde{g})$ is a Riemannian universal covering of (M,g), the deck transformations are isometries and we get $\widehat{\mathcal{S}}_k(x_1,T) = \widehat{\mathcal{S}}_k(x_2,T)$.

Let Λ_0 be the set of closed curves with trivial homotopy on M. For $k \in \mathbb{R}$, we define the action $\mathcal{S}_k : \Lambda_0 \times \mathbb{R}^+ \to \mathbb{R}$ as follow, given $(x,T) \in \Lambda_0 \times \mathbb{R}^+$ consider a lift $\widetilde{x}$ of x to $\widetilde{M}$ and take $\mathcal{S}_k(x,T) := \widehat{\mathcal{S}}_k(\widetilde{x},T)$, from above remarks it is well defined.

3 Mountain pass and Palais-Smale condition

The following appears in [1], Lemma 5.1 the same argument still working

Lemma 3.1 *Let θ be a bounded 1-form in $\widetilde{M}$ and $W \subset \widetilde{M}$ be a neighbourhood of the point x_0. Then there exists an open ball $U \subset W$ centred at x_0 and $c > 0$ such that, if γ is a closed curve in U then $\left|\int_\gamma \theta_x\right| \le c \cdot length(\gamma)^2$.*

Now, we show that for energy levels $0 = e_0 < k < c(L)$, the action functional $\mathcal{S}_k$ exhibits a mountain pass on Λ_0.

Theorem 3.2 *Let $k > e_0 = 0$. There exists $c > 0$ such that if $\Gamma : [0,1] \to \Lambda_0$, is a path joining constant curve $\Gamma(0) = x_0 : [0,T_0] \to x_0 \subset M$ to any closed curve $\Gamma(1)$ with negative (L+k)-action then $\sup_{s\in[0,1]} S_{L+k}(\Gamma(s)) > c > 0$.*

Proof: Let $\{V_i\}_{i=1}^N$ be a finite cover of M by open balls such that $\cup U_{i_\alpha} = p^{-1}V_i$ where U_{i_α} are open balls given by lemma 3.1 with the corresponding constant

c_{i_α}. Note that $c_{i_\alpha} > 0$ only depends on V_i, we call this constant c_i. Let $r_0 > 0$ be such that any ball of radius r_0 in M is contained in some V_i, write $b := \max_{1\leq i\leq N} c_i$ and set $0 < l_0 < \min\{r_0, \sqrt{\frac{k}{2b^2}}\}$.

Claim: There exists $s_0 \in [0,1]$ such that length$(\Gamma(s_0)) = l_0$.

Proof of Claim: If for some $s_1 \in [0,1], \Gamma(s_1)$ is not contained in some V_i the claim is clear, therefore suppose that $\Gamma(s)$ is contained in some V_i for all $s \in [0,1]$. Suppose that length$(\Gamma(s)) < l_0$ for all $s \in [0,1]$ as $l_0 < r_0$ then for all $s \in [0,1]$, $\Gamma(s)$ is contained in some V_i. Therefore the lifting $\widetilde{\Gamma}(s)$ is contained in some U_{i_α} for all $s \in [0,1]$. Writing $\gamma_1 := \Gamma(1) : [0, T_1] \to M$ and $l_1 :=$ length(γ_1) we have

$$0 > S_{L+k}(\gamma_1) \geq \frac{1}{2}\int_0^{T_1} |\dot{\widetilde{\gamma}}_1|^2 - |\int_{\widetilde{\gamma_1}} \theta_x| + kT_1 \geq \frac{1}{2}\int_0^{T_1} |\dot{\widetilde{\gamma}}_1|^2 - bl_1^2 + kT_1.$$

By Schwartz inequality $l_1^2 = (\int_0^{T_1} |\dot{\widetilde{\gamma}}_1|\ dt)^2 \leq T_1 \cdot \int_0^{T_1} |\dot{\widetilde{\gamma}}_1|^2\ dt$, therefore $0 > S_{L+k}(\gamma_1) \geq (\frac{1}{2T_1} - b)l_1^2 + kT_1$, but $k > 0$ and $T_1 > 0$ then $\frac{1}{2T_1} - b < 0$, i.e., $T_1 > \frac{1}{2b}$, thus $l_1^2 > \frac{kT_1}{b - \frac{1}{2T_1}} > \frac{kT_1}{b} > \frac{k}{2b^2} > l_0^2$. The claim follows from continuity of $s \to$ length$(\Gamma(s))$ and that length$(\Gamma(0)) = 0$.

We continue the proof of theorem 3.2. Take $g(t) := (\frac{1}{2t} - b)l_0^2 + kt$ since length$(\Gamma(s_0)) = l_0 < r_0$ then $\Gamma(s_0)$ is contained in some V_i. Taking a lift $\widetilde{\Gamma}(s_0)$ contained in U_{i_α} we can apply Lemma 3.1, therefore if $\Gamma : [0,1] \to \Lambda_0$ as $l_0 < \sqrt{\frac{k}{2b^2}} < \frac{\sqrt{2k}}{b}$, we have

$$S_{L+k}(\Gamma(s_0)) \geq g(T') \geq \min_{t\in\mathbb{R}^+} g(t) = l_0[\sqrt{2k} - bl_0] =: c > 0. \qquad \square$$

We will say that $\mathcal{S}_k$ satisfies the *Palais-Smale condition* on Λ_0. If every sequence (x_n, T_n) in the same connected component of Λ_0 with $|\mathcal{S}_k(x_n, T_n)|$ bounded and $\lim_n \|d_{(x_n,T_n)}\mathcal{S}_k\|_{(x_n,T_n)} = 0$ has a convergent subsequence. We say that $\mathcal{S}_k$ satisfies the Palais-Smale condition at the level a (resp. levels in $[a,b]$). If the above condition is valid restricted to (x_n, T_n) with $\lim_n \mathcal{S}_k(x_n, T_n) = a$ (resp. $lim_n \mathcal{S}_k(x_n, T_n) \in [a,b]$). In order to study the Palais-Smale condition of $\mathcal{S}_k$ we need some preliminary results, from the convexity of L result

Lemma 3.3 *Suppose that $L(x,v) = \frac{1}{2}|v|_x^2 + \theta_x(v)$. Given any coordinate chart there are positive numbers δ, B, a and D such that for x, y with $d(x,y) < \delta$ we have*

$$a|v - w|^2 \leq (L_v(x,v) - L_v(y,w)) \cdot (v - w) + [B(|v| + |w|) + D]\ |v - w|\ d(x,y).$$

Lemma 3.4 *If $x_n \in \Lambda_0, \mathcal{S}_k(x_n, T_n) < A_1$ and $T_n \leq A_2$ there is $B > 0$ with $\frac{l_n^2}{T_n} \leq \int_0^{T_n} |\dot{y}_n|^2 dt = \frac{1}{T_n}\int_0^1 |\dot{x}_n|^2 dt < B$, where $l_n :=$ length(x_n) and $y_n(t) = x_n(\frac{t}{T_n})$. In particular if $T_n \to 0$ then $\lim_n l_n = 0$.*

Proof: Take $l_n := \text{length}(y_n)$, by Schwartz inequality $l_n^2 = (\int_0^{T_n} |\dot{y}_n| dt\ )^2 \leq T_n \cdot \int_0^{T_n} |\dot{y}_n|^2 dt$, thus we obtain the first inequality. Now, since $L(x,v) = \frac{1}{2}|v|_x^2 + \theta_x(v)$ is superlinear, i.e., for all $A \in \mathbb{R}$ there is $C \in \mathbb{R}$ such that $L(x,v) \geq A|v| - C$ for all $(x,v) \in T\widetilde{M}$, using that the actions $\mathcal{S}_k(x_n, T_n)$ are bounded a calculation gives the second inequality. □

For p_1, p_2 in a boundaryless, complete manifold N and $T > 0$ take

$$\mathcal{C}_T(p_1,p_2) := \{\gamma \in \mathcal{C}^{ac}([0,T], N) \mid \gamma(0) = p_1,\ \gamma(T) = p_2\}.$$

Theorem 3.5 *([2], th. 3-1.11.) Let $\mathbb{L}$ be a convex, superlinear Lagrangian on N. Let N be boundaryless, complete Riemannian manifold then for any $p_1, p_2 \in N$, $T > 0$, $b \in \mathbb{R}$ the set $\mathcal{A}(b) := \{\gamma \in \mathcal{C}_T(p_1,p_2) \mid S_{\mathbb{L}}(\gamma) \leq b\}$, is compact in the C^0-topology.*

The next result gives some conditions which imply the Palais-Smale condition

Proposition 3.6 *If a sequence $(x_n, T_n)_{n\in\mathbb{N}} \subset \Lambda_0$ satisfies*

$$|\mathcal{S}_k(x_n,T_n)| < A_1, \quad \|d_{(x_n,T_n)}\mathcal{S}_k\| < \frac{1}{n} \quad \textit{and} \quad 0 < \liminf_n T_n < +\infty,$$

then there exists a convergent subsequence.

Proof: From compactness of M given $(x_n, T_n) \in \Lambda_0$ taking a subsequence we can assume that $q_0 := \lim_n x_n(0) = \lim_n x_n(1)$, and that $T := \lim_n T_n \in \mathbb{R}^+$ exist, also suppose that $d(x_n(0), q_0) < \delta$ for $\delta > 0$ small enough. Let $y_n(t) := x_n(\frac{t}{T_n})$ and $\alpha_n : [0,1] \to M$ be the geodesic joining $\alpha_n(0) := q_0$, $\alpha_n(1) = x_n$, and take $\beta_n : [T_n + 1, T + 3] \to M$ geodesics such that $\beta_n(T_n + 1) = y_n(T_n)$, $\beta_n(T+3) = q_0$ then $|\dot{\alpha}_n| \leq 1$ and $|\dot{\beta}_n| \leq 1$. We define

$$w_n(t) := \begin{cases} \alpha_n(t) & \text{if } 0 \leq t \leq 1, \\ y_n(t-1) & \text{if } 1 \leq t \leq T_n + 1, \\ \beta_n(t) & \text{if } T_n + 1 \leq t \leq T + 3. \end{cases}$$

All the closed curves $w_n : [0, T+3] \to M$ are based at q_0 and their actions are uniformly bounded. Now, applying theorem 3.5 with $N := \widetilde{M}$, $p_1 = p_2 = \widetilde{q}_0$ then the set w_n is relatively compact in the C^0-topology. Thus there is a convergent subsequence of w_n in the C^0 topology. From now, we will work with a convergent subsequence of $\{x_n\}$. We shall assume without loss of generality that the limit point of $\{x_n\}$ is contained in a coordinate chart and work in $\mathbb{R}^{2n}$. Taking $z_n := \frac{x_n}{T_n}$ since $0 < T < \infty$ by lemma 3.4 there is $K > 0$ with $\|x_n\|_{\mathcal{H}^1} \leq K$ and $\|z_n\|_{\mathcal{H}^1} \leq K$, as $\lim_n \|d_{(x_n,T_n)}\mathcal{S}_k\|_{\mathcal{H}^1\times\mathbb{R}} = 0$ we get

$$\left|\int_0^1 [T_n L_x(x_n, \dot{z}_n) - T_m L_x(x_m, \dot{z}_m)](x_n - x_m) + [L_v(x_n, \dot{z}_n) - L_v(x_m, \dot{z}_m)](\dot{x}_n - \dot{x}_m) ds\right|,$$

is less than $\frac{\epsilon}{4}$ for $n, m > N$. Since $L(x,v) = \frac{1}{2}|v|_x^2 + \theta_x(v)$ then there exists a constant $c > 0$ such that $\|L_x\| < c|v|_x^2$. Therefore the first term is bounded by

$[T_n + T_m](cK^2)\|x_n - x_m\|_\infty$ thus the second term is small. By lemma 3.3 and the above remark we have

$$a\int_0^1 |\dot{z}_n - \dot{z}_m|^2\, ds \leq \frac{\epsilon}{2} + B\int_0^1 (|\dot{z}_n| + |\dot{z}_m|)|\dot{z}_n - \dot{z}_m||x_n - x_m| + D|\dot{z}_n - \dot{z}_m||x_n - x_m| ds,$$

thus $a\|\dot{z}_n - \dot{z}_m\|^2_{\mathcal{L}^2} \leq \frac{\epsilon}{2} + (4BK^2 + 2KD)\|x_n - x_m\|_\infty$, as x_n converges in the C^0-topology we get that z_n converges in the $\mathcal{H}^1$-norm. Therefore x_n converges in the $\mathcal{H}^1 \times \mathbb{R}^+$-norm because $T \neq 0$ as we wished to show. □

4 Minimax method and Proof of theorem 1.1

This section follows closely Contreras' ideas in [1]. Let G be an open set in a Riemannian manifold and $f : G \to \mathbb{R}$ a C^2 function. The vector field $-\bigtriangledown f$ is not necessarily Lipschitz then the gradient flow ψ_s of $-f$ might be only a local flow. Fro $p \in G$ define $\alpha(p) := \sup\{t > 0 \mid s \to \psi_s(p)$ is defined on $s \in [0, t]\}$. We say that the flow ψ_s is *relatively complete* on $[a \leq f \leq b]$ if for $a \leq f(p) \leq b$, either $\alpha(p) = +\infty$ or $f(\psi_s(p)) \leq a$ for some $0 \leq s < \alpha(p)$.

We shall use the following mountain pass theorem

Proposition 4.1 *([1], cor. 6.5) Let G be a C^2 Riemannian manifold and $f : G \to \mathbb{R}$ a C^2 function. Let $p, q \in G$ and $c := \inf_{\Gamma \in C(p,q)} \sup_{s \in [0,1]}\{f(\Gamma(s))\}$, where $C(p, q) := \{\Gamma : [0, 1] \to G \mid \Gamma \in C^0, \Gamma(0) = p, \Gamma(1) = q\}$. Suppose that*

- *i).- $c \in \mathbb{R}$ and the gradient flow of $-f$ is relatively complete on $|f - c| \leq \epsilon$ for some $\epsilon > 0$.*
- *ii).-* $\max\{f(p), f(q)\} < c$.
- *iii).- There are closed subsets $B \subset A \subset G$ such that*
  - *a).- f satisfies the Palais-Smale condition restricted to A, at level c.*
  - *b).- For some $\epsilon_1 > 0$, A contains the ϵ_1-neighbourhood of B.*
  - *c).- For all $\epsilon > 0$, there are $\lambda \in (0, \epsilon)$ and $\Gamma \in C(p, q)$ such that $\Gamma([0, 1]) \subset (B \cap [f \leq c + \epsilon]) \cup [f \leq c - \lambda]$.*

Then c is a critical value of f moreover $K_{c,A} := \{\ x \in G \mid x \in A, df(x) = 0, f(x) = c\ \}$, contains a point which is not a strict local minimizer.

Now we analyse whether $\mathcal{S}_k$ satisfies the conditions of the previous proposition

Lemma 4.2 *For all $k \in \mathbb{R}$, if $0 \notin [a, b] \subset \mathbb{R}$, then the gradient flow of $-\mathcal{S}_k$ on Λ_0 is relatively complete on $[a \leq \mathcal{S}_k \leq b]$.*

Proof: The proof is a straightforward application to $L(x,v) = \frac{1}{2}|v|_x^2 + \theta_x(v)$ of proof of lemma 6.7 in [1]. $\square$

The next follows parallel arguments in [1], but applied to our context

Proposition 4.3 *([1], p. 7.1) Take* $c(k) := \inf_{\Gamma\in\mathcal{C}(\gamma_0,\gamma_1)} \sup_{s\in[0,1]}\{\mathcal{S}_k(\Gamma(s))\}$, *where* $\gamma_0, \gamma_1 \in \Lambda_0$, $\mathcal{C}(\gamma_0,\gamma_1) := \{\Gamma : [0,1] \to \Lambda_M \mid \Gamma \in C^0, \Gamma(0) = \gamma_0, \Gamma(1) := \gamma_1\}$ *and* $k \in \mathbb{R}$. *Suppose that for some* $k_0 \in \mathbb{R}$ *we have* $\lim_{k\to k_0^+} c(k) \neq 0$ *and that* $c(k_0) > \max\{\mathcal{S}_{k_0}(\gamma_0), \mathcal{S}_{k_0}(\gamma_1)\}$. *Then, there exists* $\epsilon > 0$ *such that for Lebesgue almost all* $k \in (k_0, k_0+\epsilon), c(k)$ *is a critical value for* $\mathcal{S}_k$ *on* Λ_0 *with a critical point which is not a strict local minimizer.*

Proof: Observe that function $k \to c(k)$ is non-decreasing. Since $\mathcal{S}_k$ is continuous on k there is $\epsilon > 0$ such that

$$\max\{\mathcal{S}_k(\gamma_0), \mathcal{S}_k(\gamma_1)\} < c(k_0) \leq c(k) \neq 0 \ \ \forall\, k_0 < k < k_0 + \epsilon. \tag{1}$$

By Lebesgue's theorem there is a total measure subset of $k_0 < k < k_0+\epsilon$, where $c(\cdot)$ is locally Lipschitz. Given k in this set and M_k its Lipschitz constant. Let $B \subset A \subset \Lambda_0$ be the closed subset defined by $B := \{(x,T) \in \Lambda_0 \mid T \leq M_k + 2\}$ and $A := \{(x,T) \in \Lambda_0 \mid T \leq M_k + 3\}$. From (1), $c(k) \neq 0$ by prop. 3.6 we get that $\mathcal{S}_k$ satisfies the Palais-Smale condition restricted to A at level $c(k)$.

Claim: There exists $\Gamma \in \mathcal{C}(\gamma_0,\gamma_1)$, $\epsilon > 0$ and $\lambda > 0$ such that

$$\Gamma([0,1]) \subset (B \cup (\mathcal{S}_k \leq c(k) - \lambda)) \cap (\mathcal{S}_k \leq c(k) + \epsilon).$$

Assuming the Claim, from proposition 4.1 we have that $\mathcal{S}_k$ has a critical point in A which is not a strict local minimizer this prove the proposition 4.3.

Proof of Claim: Fix k such that $c(\cdot)$ is locally Lipschitz in k we take a sequence $k_n \geq k$ with $k_n \to k$, then $\mathcal{S}_{k_n}, \mathcal{S}_k$ have a mountain pass geometry with the same set of paths in $\mathcal{C}(\gamma_0,\gamma_1)$. Let $\Gamma_n \in \mathcal{C}(\gamma_0,\gamma_1)$ be a path such that

$$\max_{s\in[0,1]} \mathcal{S}_{k_n}(\Gamma_n(s)) \leq c(k_n) + (k_n - k).$$

Now, since $c(\cdot)$ is continuous in k and $k \to \mathcal{S}_k(\gamma)$ is increasing then

$$\max_{s\in[0,1]} \mathcal{S}_k(\Gamma_n(s)) \leq \max_{s\in[0,1]} \mathcal{S}_{k_n}(\Gamma_n(s)) \leq c(k_n) + (k_n - k) \to c(k).$$

Take $s \in [0,1]$ with $\mathcal{S}_k(\Gamma_n(s)) > c(k) + (k_n - k)$ then $\Gamma_n(s) = (x,T)$ where

$$T := \frac{\mathcal{S}_{k_n}(\Gamma_n(s)) - \mathcal{S}_k(\Gamma_n(s))}{k_n - k} \leq M_k + 2,$$

for n enough large. Given $\epsilon > 0$ take n with $c(k_n) - c(k) + (k_n - k) < \epsilon$ and $\lambda := k_n - k$ then $\Gamma([0,1]) \subset (B \cup (\mathcal{S}_k \leq c(k) - \lambda)) \cap (\mathcal{S}_k \leq c(k) + \epsilon)$, which proves the Claim. $\square$

Proof of theorem 1.1: If $0 = e_0 < k < c(L)$ by definition of $c(L)$ there is $(x_1, T_1) \in \Lambda_0$ such that $\mathcal{S}_k(x_1, T_1) < 0$. Now taking a constant curve (x_0, T_0), from theorem 3.2 there exists a mountain pass geometry when we consider families of curves going from a constant curve with arbitrarily small action (x_0, T_0) to a curve with negative action (x_1, T_1). From proposition 4.3 there is $\epsilon > 0$ such that for almost all $k \in (k, k+\epsilon)$ there is a critical point of $\mathcal{S}_k$ in Λ_0, this is a periodic orbit with trivial homotopy and positive $(L+k)$-action. $\square$

References

[1] G. Contreras. The Palais-Smale condition on contact type energy levels for convex Lagrangian systems, *Cal. Var.*, 27(3) (2006), 321-395.

[2] G. Contreras, R. Iturriaga. Global minimizers of autonomous lagrangians *22 Col. Brasileiro de Mat.*, IMPA, (1999).

[3] U. F. Schlenk, applications of Hofer's geometry to Hamiltonian dynamics, *Comm. Math. Helv.*, 81, (2006), 105-121.

[4] L. Macarini, Hofer-Zehnder capacity and Hamiltonian circle actions, *Comm. in Contemp. Math.*, 6 (6) (2004), 918-945.

[5] Steve Smale, Morse theory and a non-linear generalization of the Dirichlet problem, *Ann. of Math.* 80 (1964), 382-396.

Received: March 12, 2013