Differential Equation of K -Bessel's Function and its Properties

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Abstract

In this paper we solve a differential equation for K - Bessel function. We establish a relationship between Bessel function and K -Bessel function. Finally we evaluate the generating function for K- Bessel function.

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1 Introduction

In [1] the authors introduce the generalized K-Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k > 0, x \in C \setminus kZ^-, \quad (1)$$

where $(x)_{n,k}$ is the K-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k).....(x+(n-1)k), x \in C, k \in R, n \in N^+. \quad (2)$$

The integral form of the generalized K-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, x \in C, k \in R, \operatorname{Re}(x) > 0, \quad (3)$$

from which it follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right). \quad (4)$$

The Bessel Function (cf. [2], page 109), given as

$$J_\vartheta(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r+\vartheta}}{\Gamma(r+\vartheta+1)(r!)}. \quad (5)$$

If ϑ is a negative integer, then

$$J_{-\vartheta}(z) = (-1)^\vartheta J_\vartheta(z). \quad (6)$$

2 Main result

In this section we introduce a differential equation known as K- Bessel differential equation and evaluate its solution which we call as the K- Bessel function. In theorem 2, we evaluate the functional relation between K -Bessel function and Bessel function. Finally in theorem 4, we find the generating function for K- Bessel function.

Theorem 1. The K- Bessel Differential equation is defined as,

$$\frac{d^2 y}{dz^2} + \frac{1}{z} \frac{dy}{dz} + \frac{1}{k^2} \left(k - \frac{\vartheta^2}{z^2}\right) y = 0. \quad (7)$$

Its solution is known as, K- Bessel function and it is written as,

$$J_{\pm\vartheta}^k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r \pm \frac{\vartheta}{k}}}{\Gamma_k(rk \pm \vartheta + k)(r!)}, \quad (8)$$

where $k \in R^+$, $\vartheta \in I$ and $\vartheta > -k$.

Proof. We can express the solution of (7), in the form of a series of ascending powers of z . Let us assume that its series solution is

$$y = \sum_{r=0}^{\infty} a_r z^{m+r}, \quad (9)$$

from which we get

$$\frac{dy}{dz} = \sum_{r=0}^{\infty} a_r(m+r)z^{m+r-1}, \quad (10)$$

and

$$\frac{d^2y}{dz^2} = \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)z^{m+r-2}. \quad (11)$$

Substituting Eqs. (9)-(11) into Eq. (7), we get

$$\sum_{r=0}^{\infty} a_r \left[\{(m+r)^2 - \frac{\vartheta^2}{k^2}\} z^{m+r-2} + \frac{1}{k} z^{m+r} \right] = 0. \quad (12)$$

Since relation (12) is an identity, the coefficients of various powers of z must be zero. Equating to zero the coefficient of lowest powers of z , i.e. z^{m-2} in (12), we have

$$a_0 \neq 0, \Rightarrow m = \pm \frac{\vartheta}{k}. \quad (13)$$

Now equating to zero the coefficient of z^{m-1} in (12), we have

$$a_1 \left\{ (m+1)^2 - \frac{\vartheta^2}{k^2} \right\} = 0,$$

But $(m+1)^2 - \frac{\vartheta^2}{k^2} \neq 0$, since $m = \pm \frac{\vartheta}{k}$, from (13), $\therefore a_1 = 0$.

Again equating to zero the coefficient of general term, i.e. z^{m+r} in (12), we have

$$a_{r+2} = \frac{-a_r}{k(m+r+2 - \frac{\vartheta}{k})(m+r+2 + \frac{\vartheta}{k})}. \quad (14)$$

Putting $r = 1, 3, 5, \dots$ etc in (14), we have $a_1 = a_3 = \dots = 0$ (each).

When $m = \frac{\vartheta}{k}$ from (14), we have

$$a_{r+2} = \frac{-a_r}{k(r+2)(r+2 + \frac{2\vartheta}{k})}. \quad (15)$$

Putting $r = 2, 4, 6, \dots$ etc in (15), we have

$$a_2 = \frac{-a_0}{(2\sqrt{k})^2(1!)(\frac{\vartheta}{k} + 1)},$$

$$a_4 = \frac{(-1)^2 a_0}{(2\sqrt{k})^4 (2!) (\frac{\vartheta}{k} + 1) (\frac{\vartheta}{k} + 2)},$$

$$a_6 = \frac{(-1)^3 a_0}{(2\sqrt{k})^6 (3!) (\frac{\vartheta}{k} + 1) (\frac{\vartheta}{k} + 2) (\frac{\vartheta}{k} + 3)}, \text{ etc.}$$

put the values of $a_1, a_2, a_3, \dots$ etc in (9), we have

$$y = a_0 (2\sqrt{k})^{\frac{\vartheta}{k}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2\sqrt{k}}\right)^{2r + \frac{\vartheta}{k}}}{(r!) (\frac{\vartheta}{k} + 1) (\frac{\vartheta}{k} + 2) \dots (\frac{\vartheta}{k} + r)}.$$

If $a_0 = \frac{1}{(2k)^{\frac{\vartheta}{k}} \Gamma(\frac{\vartheta}{k} + r)}$ then, we have

$$J_{\vartheta}^k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r + \frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k) (r!)}.$$

And for $m = -\frac{\vartheta}{k}$, similarly we have

$$J_{-\vartheta}^k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r - \frac{\vartheta}{k}}}{\Gamma_k(rk - \vartheta + k) (r!)}.$$

Hence.

Theorem 2. The functional relation between K-Bessel function and Bessel function is given by

$$J_{\vartheta}^k(z) = k^{-\frac{\vartheta}{2k}} J_{\frac{\vartheta}{k}}\left(\frac{z}{\sqrt{k}}\right) \quad (16)$$

or the counterpart is

$$J_{\frac{\vartheta}{k}}(z) = k^{\frac{\vartheta}{2k}} J_{\vartheta}^k(z\sqrt{k}) \quad (17)$$

and another form

$$J_{\vartheta k}^k(z) = k^{-\frac{\vartheta}{2}} J_{\vartheta}\left(\frac{z}{\sqrt{k}}\right) \quad (18)$$

or the counterpart is

$$J_{\vartheta}(z) = k^{\frac{\vartheta}{2}} J_{\vartheta k}^k(z\sqrt{k}) \quad (19)$$

Proof. From definition of K - Bessel function (8), consider the positive sign,

$$J_{\vartheta}^k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)},$$

using (4) and rearranging the terms we have

$$J_{\vartheta}^k(z) = k^{-\frac{\vartheta}{2k}} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2\sqrt{k}}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma\left(r + \frac{\vartheta}{k} + 1\right)(r!)},$$

$$J_{\vartheta}^k(z) = k^{-\frac{\vartheta}{2k}} J_{\frac{\vartheta}{k}}\left(\frac{z}{\sqrt{k}}\right).$$

Hence.

Similarly we can prove (17),(18) and (19).

Theorem 3. If ϑ is a negative integer such that $\vartheta = mk$, where m is an integer, then

$$J_{-\vartheta}^k(z) = (-k)^{\frac{\vartheta}{k}} J_{\vartheta}^k(z) \quad (20)$$

Proof. From (16) we have

$$J_{-\vartheta}^k(z) = k^{\frac{\vartheta}{2k}} J_{-\frac{\vartheta}{k}}\left(\frac{z}{\sqrt{k}}\right),$$

using (6), we obtain

$$J_{-\vartheta}^k(z) = (-1)^{\frac{\vartheta}{k}} k^{\frac{\vartheta}{2k}} J_{\frac{\vartheta}{k}}\left(\frac{z}{\sqrt{k}}\right),$$

$$J_{-\vartheta}^k(z) = (-1)^{\frac{\vartheta}{k}} k^{\frac{\vartheta}{k}} k^{-\frac{\vartheta}{2k}} J_{\frac{\vartheta}{k}}\left(\frac{z}{\sqrt{k}}\right),$$

using (16), we have

$$J_{-\vartheta}^k(z) = (-k)^{\frac{\vartheta}{k}} J_{\vartheta}^k(z).$$

Hence.

Theorem 4.(Generating Function for K - Bessel Function) For $x \neq 0$ and for all finite z ,

$$e^{\frac{z}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}} - \frac{\sqrt{k}}{x}\right)} = \sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J_{\vartheta k}^k(z) \quad (21)$$

Proof. Consider the right side of (21),

$$A \equiv \sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J_{\vartheta k}^k(z),$$

let us collect powers of z in the summation

$$A \equiv \sum_{\vartheta=-\infty}^{-1} x^{\vartheta} J_{\vartheta k}^k(z) + \sum_{\vartheta=0}^{\infty} x^{\vartheta} J_{\vartheta k}^k(z),$$

$$A \equiv \sum_{\vartheta=0}^{\infty} x^{-\vartheta-1} J_{-(\vartheta+1)k}^k(z) + \sum_{\vartheta=0}^{\infty} x^{\vartheta} J_{\vartheta k}^k(z),$$

using relation (20), we have

$$A \equiv \sum_{\vartheta=0}^{\infty} x^{-\vartheta-1} (-1)^{\vartheta+1} k^{\vartheta+1} J_{(\vartheta+1)k}^k(z) + \sum_{\vartheta=0}^{\infty} x^{\vartheta} J_{\vartheta k}^k(z),$$

using (8), we have

$$A \equiv \sum_{\vartheta=0}^{\infty} \sum_{r=0}^{\infty} \frac{x^{-\vartheta-1} (-1)^{\vartheta+r+1} k^{\vartheta+1} \left(\frac{z}{2}\right)^{2r+\vartheta+1}}{(r!) \Gamma_k(rk + (\vartheta+1)k + k)} + \sum_{\vartheta=0}^{\infty} \sum_{r=0}^{\infty} \frac{x^{\vartheta} (-1)^r \left(\frac{z}{2}\right)^{2r+\vartheta}}{(r!) \Gamma_k(rk + \vartheta k + k)},$$

For elementary series manipulation (cf. [2], equation 12, page 58), we have

$$A \equiv \sum_{\vartheta=0}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta}{2} \rfloor} \frac{x^{-\vartheta+2r-1} (-1)^{\vartheta-r+1} k^{\vartheta-2r+1} \left(\frac{z}{2}\right)^{\vartheta+1}}{(r!) \Gamma_k(rk + (\vartheta-2r+1)k + k)} + \sum_{\vartheta=0}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta}{2} \rfloor} \frac{x^{\vartheta-2r} (-1)^r \left(\frac{z}{2}\right)^{\vartheta}}{(r!) \Gamma_k(rk + (\vartheta-2r)k + k)},$$

$$A \equiv \sum_{\vartheta=1}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta-1}{2} \rfloor} \frac{x^{-\vartheta+2r} (-1)^{\vartheta-r} k^{\vartheta-2r} \left(\frac{z}{2}\right)^{\vartheta}}{(r!) \Gamma_k(-rk + \vartheta k + k)} + 1 + \sum_{\vartheta=1}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta}{2} \rfloor} \frac{x^{\vartheta-2r} (-1)^r \left(\frac{z}{2}\right)^{\vartheta}}{(r!) \Gamma_k(-rk + \vartheta k + k)},$$

using (4) and rearranging the terms

$$A \equiv 1 + \sum_{\vartheta=1}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta}{2} \rfloor} \frac{(-1)^r \left(\frac{x}{\sqrt{k}}\right)^{\vartheta-r-r} \left(\frac{z}{2\sqrt{k}}\right)^{\vartheta}}{(r!) (\vartheta-r)!} + \sum_{\vartheta=1}^{\infty} \sum_{r=0}^{\lfloor \frac{\vartheta-1}{2} \rfloor} \frac{(-1)^{\vartheta-r} \left(\frac{x}{\sqrt{k}}\right)^{r-(\vartheta-r)} \left(\frac{z}{2\sqrt{k}}\right)^{\vartheta}}{(r!) (\vartheta-r)!},$$

using ([2], Lemma 12, Page 112), to conclude that

$$A \equiv 1 + \sum_{\vartheta=1}^{\infty} \sum_{r=0}^{\vartheta} \frac{(-1)^r \left(\frac{x}{\sqrt{k}}\right)^{\vartheta-r-r} \left(\frac{z}{2\sqrt{k}}\right)^{\vartheta}}{(r!) (\vartheta-r)!},$$

$$A \equiv \sum_{\vartheta=0}^{\infty} \sum_{r=0}^{\vartheta} \frac{(-1)^r \left(\frac{x}{\sqrt{k}}\right)^{\vartheta-r} \left(\frac{\sqrt{k}}{x}\right)^r \left(\frac{z}{2\sqrt{k}}\right)^{\vartheta}}{(r!)(\vartheta-r)!},$$

$$A \equiv \sum_{\vartheta=0}^{\infty} \frac{\left(\frac{x}{\sqrt{k}} - \frac{\sqrt{k}}{x}\right)^{\vartheta} \left(\frac{z}{2\sqrt{k}}\right)^{\vartheta}}{(\vartheta!)},$$

$$A \equiv \exp\left[\frac{z}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}} - \frac{\sqrt{k}}{x}\right)\right].$$

Hence.

Particular Cases

[A] Put $k = 1$ in (7), we have well known Bessel's differential equation given by ([2], page 109) and (8) reduces into well known Bessel's function given by ([2], page 109).

[B] Put $k = 1$ in (20), it will reduce into known result given by ([2], page 109).

[C] Put $k = 1$ in (21), it will reduce into known result given by ([2], page 113).

Reference

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