An Abstract Approach to Nonlinear Boundary Value Problems at Resonance

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Abstract

In this paper we provide an abstract existence result for nonlinear boundary value problems at resonance. Our abstract approach provides a general framework for the situation that occurs for a large class of differential operators. We will approach the problem using an Alternative Method in conjunction with topological degree theory.

Mathematics Subject Classification: 34B15

Keywords: Boundary Value Problems, Resonance, Lyapunov-Schmidt, Topological degree theory

1 Introduction

In, [1], Cesari and Kannan prove an abstract existence theorem for a class of “selfadjoint” differential operators. Throughout they assume the nonlinearity involved is bounded. In this paper we extend the abstract existence result proved in [1] to a more general “non-selfadjoint” case in which the nonlinearity is also allowed to be unbounded.

2 Preliminaries

Let S denote an arbitrary Hilbert space. We will use $\langle \cdot, \cdot \rangle$ to denote the inner product on S and $\|\cdot\|$ to denote the norm induced by this inner product. We will also use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$.

Let $\mathcal{L} : \text{dom}(\mathcal{L}) \subset S \rightarrow S$ be a linear map which satisfies $0 < \dim(\ker(\mathcal{L})) = \dim(\text{coker}(\mathcal{L})) < \infty$, where $\text{coker}(\mathcal{L})$ denotes $S/\text{Im}(\mathcal{L})$. We let $P : S \rightarrow S$ denote the orthogonal projection of S onto $\ker(\mathcal{L})$ and $E : S \rightarrow S$ the orthogonal projection of S onto $\text{Im}(\mathcal{L})$. It is trivial that $\mathcal{L}$ restricted to $\text{Ker}(P) \cap \text{dom}(\mathcal{L})$ is one-to-one and onto $\text{Im}(\mathcal{L})$. We let M_p denote the inverse of $\mathcal{L}|_{\text{Ker}(P) \cap \text{dom}(\mathcal{L})}$. We will assume that M_p is a compact linear operator on S .

We would like to remark that the properties of $\mathcal{L}$ listed above are a natural abstraction of the properties which occur in practice for a large class of differential operators. A special case of particular interest is the class of selfadjoint linear operators which occur in the study of Sturm-Liouville boundary value problems. For those readers interested in study of nonlinear boundary value problems with concrete instances of the differential operators described above, we suggest [4, 5, 6, 7, 8, 9, 11]

Let $\mathcal{F} : S \rightarrow S$ be a nonlinear continuous operator. We will be concerned with the existence of solutions to the operator equation

$$\mathcal{L}x = \mathcal{F}(x). \quad (1)$$

Our approach will be to study the operator equation, (1), using the following system of equations

$$\begin{cases} (I - P)x - M_p E \mathcal{F}(x) = 0 \\ \text{and} \\ (I - E)\mathcal{F}(x) = 0 \end{cases}. \quad (2)$$

The construction of this equivalent system is often referred to as the Lyapunov-Schmidt procedure. Readers interested in the details of this procedure or more general Alternative Methods may consult [2, 3, 6, 10].

If we let $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ denote orthonormal bases for $\ker(\mathcal{L})$ and $\text{Im}(\mathcal{L})^\perp$ respectively, then (2) is equivalent to

$$\begin{cases} u - M_p E \mathcal{F}(cv + u) = 0 \\ \text{and} \\ \langle \mathcal{F}(cv + u), w \rangle = 0 \end{cases}, \quad (3)$$

where $c \in \mathbb{R}^n$, $u \in \text{Im}(I - P)$, cv denotes $\sum_{i=1}^n c_i v_i$ and $\langle \mathcal{F}(cv + u), w \rangle$ is notation

for

$$\begin{bmatrix} \langle \mathcal{F}(cv + u), w_1 \rangle \\ \langle \mathcal{F}(cv + u), w_2 \rangle \\ \vdots \\ \langle \mathcal{F}(cv + u), w_n \rangle \end{bmatrix}.$$

We define an operator $H : Im(I - P) \times \mathbb{R}^n \rightarrow Im(I - P) \times \mathbb{R}^n$ by

$$H(u, c) = \begin{bmatrix} M_p E \mathcal{F}(cv + u) \\ c - \langle \mathcal{F}(cv + u), w \rangle \end{bmatrix}.$$

From our discussion above, solving (1) is equivalent to finding the fixed points of H .

3 Main Results

In, [1], Cesari and Kannan prove the existence of solutions to (1), in the “selfadjoint” case, under the following two assumptions:

1. There is a constant J_0 such that $\|\mathcal{F}(x)\| \leq J_0$ for each $x \in S$.
2. There is a constant $R_0 \geq 0$ such that for all $x \in S$ with $\|Px\| \geq R_0$ we have $\langle \mathcal{F}(x), Px \rangle \geq 0$ (or $\langle \mathcal{F}(x), Px \rangle \leq 0$), whenever $\|(I - P)x\| \leq \|M_p E\| J_0$.

In what follows, we extend these ideas to the case when $\mathcal{L}$ is not necessarily “selfadjoint”. We also allow $\mathcal{F}$ to be unbounded, but assume it must satisfy a sublinear growth condition.

We start with a lemma which will play a significant role in our proof of the main result.

Lemma 3.1. *Suppose $\mathcal{F}$ is sublinear, i.e. $\|\mathcal{F}(x)\| \leq M_1 \|x\|^\beta + M_2$ where $0 \leq \beta < 1$ and $M_1 > 0$. We then have that there exists positive constants, C_1 and C_2 , such that for each pair $(u, c) \in Im(I - P) \times \mathbb{R}^n$ satisfying $u = M_p E \mathcal{F}(cv + u)$ and $|c| \geq C_1$,*

$$\|u\| \leq C_2 \|cv\|^\beta.$$

Proof.

$$\begin{aligned}
\|u\| &\leq \|M_p E\| (M_1(\|cv\| + \|u\|)^\beta + M_2) \\
&= \|M_p E\| M_1 \|cv\|^\beta \left(1 + \frac{\|u\|}{\|cv\|}\right)^\beta + \|M_p E\| M_2 \\
&\leq \|M_p E\| M_1 \|cv\|^\beta \left(1 + \beta \frac{\|u\|}{\|cv\|}\right) + \|M_p E\| M_2.
\end{aligned}$$

Thus,

$$\frac{\|u\|}{\|cv\|^\beta} - \|M_p E\| M_1 \beta \frac{\|u\|}{\|cv\|} \leq \|M_p E\| M_1 + \frac{\|M_p E\| M_2}{\|cv\|^\beta}$$

or

$$\frac{\|u\|}{\|cv\|^\beta} \left(1 - \frac{\|M_p E\| M_1 \beta}{\|cv\|^{1-\beta}}\right) \leq \|M_p E\| M_1 + \frac{\|M_p E\| M_2}{\|cv\|^\beta}.$$

Now fix γ , with $0 < \gamma < 1$. If $|c| \geq \left(\frac{\|M_p E\| M_1 \beta}{\gamma}\right)^{\frac{1}{1-\beta}}$, then

$$\frac{\|u\|}{\|cv\|^\beta} \leq \frac{\|M_p E\| M_1}{1 - \gamma} + \frac{\|M_p E\| M_2}{(1 - \gamma) \left|\frac{\|M_p E\| M_1 \beta}{\gamma}\right|^{\frac{\beta}{1-\beta}}}$$

The result now follows by taking

$$C_1 = \left(\frac{\|M_p E\| M_1 \beta}{\gamma}\right)^{\frac{1}{1-\beta}} \text{ and } C_2 = \frac{\|M_p E\| M_1}{1 - \gamma} + \frac{\|M_p E\| M_2}{(1 - \gamma) \left|\frac{\|M_p E\| M_1 \beta}{\gamma}\right|^{\frac{\beta}{1-\beta}}}.$$

□

Remark 3.2. We will not be concerned with the case of bounded $\mathcal{F}$, this is the main reason for assuming $M_1 > 0$ in Lemma 3.1. We would, however, like to point out that in the case of bounded $\mathcal{F}$, we get the more desirable result:

If $(u, c) \in \text{Im}(I - P) \times \mathbb{R}^n$ satisfying $u = M_p E \mathcal{F}(cv + u)$, then there exists a J_0 such that

$$\|u\| \leq \|M_p E\| J_0.$$

The proof of the existence theorem found in [1] relies indispensably on this observation.

We now proceed to the proof of the main result. The following observation will be used in what follows: there exists r^* such that for all $r \geq r^*$, we have

$$r - \|M_p E\| (M_1 2^\beta r^\beta + M_2) > 0.$$

We will assume that r^* has been chosen such that C_1 (Lemma 3.1) satisfies $r^* > C_1$.

Theorem 3.3. *Suppose the following conditions hold:*

C1. $\mathcal{F}$ is “sublinear”; that is, there exist real numbers M_1, M_2 and β , with $0 \leq \beta < 1$ and $M_1 > 0$, such that $\|\mathcal{F}(x)\| \leq M_1 |x|^\beta + M_2$.

C2. There is a constant $R_0 \geq r^$ such that for all $x = cv + u \in S$ with $|c| \geq R_0$ we have $\langle \mathcal{F}(x), cv \rangle \geq 0$ (or $\langle \mathcal{F}(x), cv \rangle \leq 0$), whenever $\|u\| \leq C_2 R_0^\beta$. (Here C_2 is as in Lemma 3.1).*

Then the operator equation, (1), has at least one solution.

Proof. We will assume at first that $\langle \mathcal{F}(x), cv \rangle \geq 0$. In order to apply a degree theoretic argument, we make $Im(I - P) \times \mathbb{R}^n$ a Banach space using the following norm

$$\|(u, c)\| = \max \{\|u\|, |c|\}.$$

We then define an operator $H : Im(I - P) \times \mathbb{R}^n \rightarrow Im(I - P) \times \mathbb{R}^n$ by

$$H(u, c) = \begin{bmatrix} M_p E \mathcal{F}(cv + u) \\ c - \langle \mathcal{F}(cv + u), w \rangle \end{bmatrix}.$$

We know, from our earlier discussion, that the fixed points of H are precisely the solutions of the operator problem (1). Since M_p is compact and $\mathcal{F}$ is continuous and sublinear, we have that H is a nonlinear compact map. We will show that H has a fixed point by showing that for an appropriately chosen set, the Leray-Schauder degree of $I - H$ is nonzero.

We define

$$\Omega = B(0, R_0) \text{ in } Im(I - P) \times \mathbb{R}^n,$$

and

$$Q : [0, 1] \times \bar{\Omega} \rightarrow Im(I - P) \times \mathbb{R}^n$$

by

$$Q(\lambda, (u, c)) = \begin{bmatrix} u - \lambda M_p E F(cv + u) \\ (1 - \lambda)c + \lambda \langle \mathcal{F}(cv + u), w \rangle \end{bmatrix}.$$

It is clear that Q is a homotopy between I and $I - H$. We will analyze Q on the boundary of Ω which we denote by $\partial(\Omega)$.

It is clear that if $(u, c) \in \partial(\Omega)$, then one of two following conditions is true:

i. $\|x\| = R_0$ and $|c| \leq R_0$.

ii. $\|x\| \leq R_0$ and $|c| = R_0$.

We first look at the case where $(x, \alpha) \in \partial(\Omega)$ and i. holds.

Since $R_0 > r^*$, we have

$$R_0 - \|M_p E\| (M_1 2^\beta R_0^\beta + M_2) > 0.$$

Now,

$$\begin{aligned} \|cv + u\| &\leq \|cv\| + \|u\| \\ &= |c| + \|u\| \\ &\leq 2R_0 \end{aligned}$$

so that

$$\begin{aligned} \|M_p E \mathcal{F}(cv + u)\| &\leq \|M_p E\| (M_1 (2R_0)^\beta + M_2) \\ &< R_0 = \|u\|. \end{aligned} \tag{4}$$

We now assume $(u, c) \in \partial(\Omega)$ and that ii. holds. We may assume, without loss of generality, that $u = \beta M_p E \mathcal{F}(cv + u)$, for some $\beta \in (0, 1)$.

Now,

$$\begin{aligned} |(1 - \lambda)c + \lambda \langle \mathcal{F}(cv + u), w \rangle|^2 &= \\ (1 - \lambda)^2 |c|^2 + \lambda^2 |\langle \mathcal{F}(cv + u), w \rangle|^2 &+ 2(1 - \lambda)\lambda \langle c, \langle \mathcal{F}(cv + u), w \rangle \rangle_{\mathbb{R}}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, denotes the standard Euclidean inner product.

However,

$$\begin{aligned}\langle c, \langle \mathcal{F}(cv + u), w \rangle \rangle_{\mathbb{R}} &= \sum_{i=1}^n c_i \langle \mathcal{F}(cv + u), w_i \rangle \\ &= \left\langle \mathcal{F}(cv + u), \sum_{i=1}^n c_i w_i \right\rangle \\ &= \langle \mathcal{F}(cv + u), cw \rangle.\end{aligned}$$

Since

$$\|cv\| = |c| = R_0 \geq r^* > C_1 \text{ and } u = \beta M_p E \mathcal{F}(cv + u),$$

we have

$$\|u\| \leq C_2 \|cv\|^\beta \leq C_2 R_0^\beta.$$

It follows that $\langle \mathcal{F}(cv + u), cw \rangle \geq 0$ and thus

$$|(1 - \lambda)c + \lambda \langle \mathcal{F}(cv + u), w \rangle|^2 > 0. \quad (5)$$

The proof is now complete, from homotopy invariance of the Leray-Schauder degree, by noticing that (4) and (5) show that $Q(\lambda, (u, c))$ is nonzero for each $(u, c) \in \partial(\Omega)$ and every $\lambda \in (0, 1)$.

The case where $\langle \mathcal{F}(cv + u), cw \rangle \leq 0$ is handled in a similar manner by constructing a homotopy between $-I$ and $I - H$.

□

Remark 3.4. If we assume $\mathcal{L}$ is “self-adjoint”, then it is clear that Theorem 3.3 is a natural generalization, to the case of an unbounded nonlinearity $\mathcal{F}$, of the result found in [1].

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Received: February 18, 2014