Investigation of the Resolvent of Equation of Second Order with Normal Operator Coefficients on the Semi-Axis

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1. INTRODUCTION

Consider differential operator L in the space $H_1 = L_2[[0, \infty), H]$, where H is a separable Hilbert space generated with the expression

$$l(y) = -(P(x)y')' + Q(x)y \quad (0 \leq x < \infty) \quad (1)$$

and boundary conditions

$$y'(0) - hy(0) = 0 \quad (2)$$

here $y \in H_1$, derivatives accepted as strong, for each fixed x $P(x)$ is a uniformly differentiable bounded operator on H , generally speaking, h is a unbounded self-adjoint operator on H and $h \cdot P^{-\frac{1}{2}}(x) = P^{-\frac{1}{2}}(x) \cdot h$.

In this paper Green's function of the problems (1) - (2) is investigated. Note that, in $P(x) \equiv E$ and $h = 0$, Green's function for self adjoint Sturm Liouville equation was first studied [1] and in [8], respectively, in the case when $Q(x)$ for each x is a normal

operator on H was investigated in [2], [3]. Green's function and asymptotic behavior of the eigenvalues of the problems (1)-(2) in the self-adjoint case was studied [4] and [7]. The essential difference of this work from previous work [8] is the presence in the boundary condition an unbounded operator h , and $Q(x)$ is a normal operator.

Let's specify basic boundary conditions for operator functions $P(x), Q(x)$ and operator h , in which we'll investigate Green's function of the problems (1)-(2).

1) The operator function $P(x)$ is uniformly everywhere differentiable for all $f \in H$

$$m(f, f) \leq (P(x)f, f) \leq M \cdot (f, f), M > 0$$

2) The operators $Q(x)$ are normal and lower bounded for all $x \in [0, \infty)$ on H . The operator $Q(x)$ for all $x \in [0, \infty)$ is the inverse of completely continuous operator and its eigenvalues are laid on complex plane out of sector $\Lambda_0 = \{\lambda : |\arg \lambda - \pi| < \varepsilon_0\}$, where ε_0 is constant and $0 < \varepsilon_0 < \pi$.

3) There exist the positive constants A and $0 < a < 3/2$ such that, for all $x \in [0, \infty)$ and $|\xi - x| \leq 1$ the following inequality is true

$$\|Q^a(x) [Q(\xi) - Q(x)]\| < A |x - \xi|$$

4) There exist constant B such that, for $|\xi - x| > 1$ and for all $x \in [0, \infty)$ the inequality

$$\left\| K(\xi) \exp \left\{ -\frac{1}{2} |x - \xi| \cdot \omega \right\} \right\|_H < B$$

is fulfilled, where

$$K(x) = P^{-\frac{1}{2}}(x)Q(x)P^{-\frac{1}{2}}, \quad \omega = \{K(x) + \mu P^{-1}(x)\}^{1/2}.$$

5) For all x and ξ from $[0, \infty)$ the inequalities

$$\left\| Q(x)P^{\pm\frac{1}{2}}(x)Q^{-1}(x) \right\| < C, \quad \left\| Q(\xi)P^{-\frac{1}{2}}(x)P^{\frac{1}{2}}(\xi)Q^{-1}(\xi) \right\| < C$$

$$\|(\omega + h)^{-1}\| < C, \quad \|(\omega + h)^{-1}\omega\| < C, \quad \|e^{-\omega x}(\omega + h)^{-1}e^{\omega x}\| < C$$

are fulfilled.

6) Since $Q(x)$ for all x is the inverse of completely continuous operator, then $K(x) = P^{-\frac{1}{2}}(x)Q(x)P^{-\frac{1}{2}}$ for all x is also inverse of completely continuous operator. Let denote by $|\beta_1(x)| \leq |\beta_2(x)| \leq \dots \leq |\beta_n(x)| \leq \dots$ the absolute values of eigenvalues of the operator $K(x)$, which we'll assume as measurable function. Then suppose that for all x series $\sum_{i=1}^{\infty} |\beta_i(x)|^{-3/2}$ is convergent and its sum $F(x)$ belong to class $L_1[0, \infty)$.

2. MAIN RESULTS

The following theorem is the main result of this work.

Theorem. If the basic boundary conditions 1)-6) are fulfilled, then for sufficiently large $\mu > 0$ there exist an inverse operator $R_\mu = (L + \mu E)^{-1}$ being an integral operator the operator kernel $G(x, \eta; \mu)$ which will be called the Green (operator) function of the operator L . $G(x, \eta; \mu)$ is an operator on H that depends on two variables x and η ($0 \leq x, \eta < \infty$) and parameter μ and satisfies the conditions:

- a) $G(x, \eta; \mu)$ is strongly continuous in variables (x, η) ;
- b) there exists a strong derivative $\frac{\partial G}{\partial \eta}$, though

$$\frac{\partial G}{\partial \eta}(x, x+0; \mu) - \frac{\partial G}{\partial \eta}(x, x-0; \mu) = -P^{-1}(x) \quad (3)$$

c)

$$-(G_\eta^1 \cdot P(\eta))_\eta^1 + G\{Q(\eta) + |\mu| E\} = 0, \quad (\eta \neq x) \quad (4)$$

$$\left(\frac{\partial G}{\partial \eta}(x; \eta; \mu) - G(x; \eta; \mu) h \right)_{\eta=0} = 0 \quad (5)$$

Proof. For proving of this theorem we refer to the paper [1]. Now deduce the following banach spaces of operator-valued functions $A(x, \eta)$ on H ($0 \leq x, \eta < \infty$), $X_1, X_2, X_3^{(p)}$ ($p \geq 1$) $X_1^{(s)}, X_2(s), X_4^{(s)}, (s > 0), X_5$ where norms correspondingly defined as follows

$$\|A(x, \eta)\|_{X_1}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)\|_H^2 d\eta \right\},$$

$$\|A(x, \eta)\|_{X_2}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)\|_2^2 d\eta \right\},$$

Here $\|A(x, \eta)\|_2$ is Hilbert-Schmidt (absolute norm) norm of operator function $A(x, \eta)$ on H .

$$\|A(x, \eta)\|_{X_3^{(p)}}^2 = \left\{ \sup_{0 < x < \infty} \int_0^\infty \|A(x, \eta)\|_H^p d\eta \right\}^{1/p}$$

$$\|A(x, \eta)\|_{X_1^{(s)}}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta) Q^s(\eta)\|_H^2 d\eta \right\}$$

$$\|A(x, \eta)\|_{X_2^{(s)}}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta) Q^s(\eta)\|_2^2 d\eta \right\}$$

$$\|A(x, \eta)\|_{X_4^{(s)}}^2 = \sup_{0 < x < \infty} \int_{-\infty}^{\infty} \|A(x, \eta) Q^s(\eta)\|_H d\eta$$

$$\|A(x, \eta)\|_{X_5}^2 = \sup_{0 < x < \infty} \sup_{0 < \eta < \infty} \|A(x, \eta)\|_H$$

Completeness of these spaces in case of $x, \eta \in (-\infty, \infty)$ was established in [1].

Let $\omega = K(x) + \mu P^{-1}(x)\mu > 0$, where $K(x) = P^{-\frac{1}{2}}(x)Q(x)P^{-\frac{1}{2}}(x)$ $X(\eta) = \text{ch}\omega\eta + \omega^{-1} \cdot \text{sh}\omega\eta$.

Consider the function

$$g(x, \eta; \mu) = \begin{cases} P^{-\frac{1}{2}}(x)X(\eta)(\omega + h)^{-1}e^{-\omega x}P^{-\frac{1}{2}}(x), & \eta \leq x \\ P^{-\frac{1}{2}}(x)e^{-\omega\eta}(\omega + h)^{-1}X(x)P^{-\frac{1}{2}}(x), & \eta \geq x \end{cases} \quad (6)$$

It is easy check that the function $g(x, \eta; \mu)$ satisfies the equation

$$P(x)\frac{\partial^2 g}{\partial \eta^2} = \{Q(x) + \mu\}g(x, \eta; \mu) \quad (7)$$

and the conditions

$$\left(\frac{\partial g}{\partial \eta} - hg\right)_{\eta=0} = 0 \quad (8)$$

$$\frac{\partial g}{\partial \eta}|_{\eta=x+0} - \frac{\partial g}{\partial \eta}|_{\eta=x-0} = -P^{-1}(x) \quad (9)$$

By using equation (6) the function $g(x, \eta; \mu)$ could be represented as follows

$$g(x, \eta; \mu) = \begin{cases} \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1}(1 + h\omega^{-1})e^{-(x-\eta)\omega}P^{-\frac{1}{2}}(x) \{1 + O(\|\omega^{-1}\|)\}, & \eta \leq x \\ \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1}(1 + h\omega^{-1})e^{-(\eta-x)\omega}P^{-\frac{1}{2}}(x) \{1 + O(\|\omega^{-1}\|)\}, & \eta > x \end{cases} \quad (10)$$

Since $\|\omega^{-1}\| = 0(1)$ uniformly on $x \in (0, \infty)$ if $\mu \rightarrow +\infty$, so it follows from (1) that

$$g(x, \eta; \mu) = \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1} \cdot e^{-|x-\eta|\omega} \cdot P^{-\frac{1}{2}}(x) [1 + 0(1)] \quad (11)$$

Consider the integral equation

$$\begin{aligned} G(x, \eta; \mu) &= g(x, \eta; \mu) - \int_0^\infty g(x, \xi; \mu) [Q(\xi) - Q(x)] G(\xi, \eta; \mu) d\xi + \\ &+ \int_0^\infty \left[\frac{1}{2}P^{-\frac{1}{2}}(x)\omega \exp(-|x - \xi|\omega) P^{-\frac{1}{2}}(x) (P(\xi) - P(x)) + g_\xi^1(\xi, \eta; \mu) P^1(\xi) G(\xi, \eta; \mu) \right] d\xi \end{aligned} \quad (12)$$

It will be shown below that for sufficiently large $\mu > 0$ the equation (12) could be solved by iteration method and this solution is the Green's operator function for operator L .

First, let's estimate norm of operator function $g(x, \eta; \mu)$. For all λ laid on complex plane out of sector Λ_0 the following inequality is true

$$|\lambda + \mu| \geq \mu \sin \varepsilon_0$$

Since

$$(\{K(x) + \mu P^{-1}(x)\} f, f) = (\{Q(x) + \mu E\} P^{-1/2}(x) f, P^{-1/2}(x) f) \geq \gamma(f, f) = (\mu - d) M^{-1}$$

$$(\{\omega^{-1} \exp(-|x - \eta| \omega)\} f, f) \leq \gamma^{-\frac{1}{2}} \exp(-|x - \eta| \gamma^{\frac{1}{2}}) \quad (13)$$

so from spectral expansion of operator $g(x, \eta; \mu)$ and inequality (13) follows that

$$\|g(x, \eta; \mu)\|_H \leq \frac{1 + 0(1)}{2m \cdot \delta_0^{1/2} \cdot \gamma^{1/2}} \exp(-|x - \eta| \delta_0^{1/2} \cdot \gamma^{1/2})$$

where $\delta_0 = \sin \varepsilon_0$.

Here and later we'll select the square root of $(\lambda + \mu)$ such that $Re(\lambda + \mu)^{1/2} > 0$.

That is why

$$\int_0^\infty \|g(x, \eta; \mu)\|_H^2 d\eta \leq \frac{(1 + 0(1))^2}{4m^2 \cdot \delta_0^{3/2} \cdot \gamma^{3/2}} \quad (14)$$

From this estimation follows that $g(x, \eta; \mu) \in X_3^{(2)}$

Denote by $\|A\|_2$ the Hilbert-Schmidt norm of the operator A on the space H .

$$\int_0^\infty \left\{ \int_0^\infty \|g(x, \eta; \mu)\|_2^2 d\eta \right\} dx \leq C \int_0^\infty \sum_{i=1}^\infty (\beta_i(x) + \mu M^{-1})^{-3/2} < \infty \quad (15)$$

For this purpose, note that for all $f \in D\{K\}$ the inequality

$$(\{K(x) + \mu M^{-1}\} f, f) \leq (\omega^2 f, f) \leq (\{K(x) + \mu m^{-1}\} f, f)$$

is fulfilled, in addition there exist the operators $\{K(x) + \mu M^{-1}\}^{-1}$, ω^2 , $\{K(x) + \mu m^{-1}\}^{-1}$ for all x and they are completely continuous. Therefore for all $g \in H$ the following inequality is true

$$(\{K(x) + \mu^{-1}\} g, g) \leq (\omega^{-2} g, g) \leq (\{K(x) + \mu M^{-1}\} g, g)$$

This implies that (see [5])

$$\lambda_j^{-2}(x) \leq (\beta_j(x) + \mu M^{-1})^{-1} \quad (16)$$

where $\lambda_j(x)$ eigenvalues of operator ω .

Then we have:

$$\|g(x, \eta; \mu)\|_2^2 \leq \frac{1}{4} \|P^{-\frac{1}{2}}(x)\| \|\omega^{-1} \cdot \exp(-|x - \eta| \omega)\|_2^2 \cdot \|P^{-\frac{1}{2}}(x)\| (1 + 0(1))$$

Form inequality (14) follows that

$$\begin{aligned} \|\omega^{-1} \cdot \exp(-|x - \eta| \omega)\|_2^2 &= \sum_{j=1}^{\infty} \lambda_j^{-2}(x) \exp(-2|x - \eta| \lambda_j(x)) \leq \\ &\leq \sum_{j=1}^{\infty} (B_j(x) + \mu M^{-1})^{-1} \exp\left\{-2|x - \eta| (B_j(x) + \mu M^{-1})^{1/2}\right\} \end{aligned}$$

Therefore we have

$$\|g(x, \eta; \mu)\|_2^2 \leq \frac{(1 + 0(1))}{4m^2} \leq \sum_{j=1}^{\infty} (B_j(x) + \mu M^{-1})^{-1} \exp\left\{|x - \eta| (B_j(x) + \mu M^{-1})^{1/2}\right\}$$

Hence by considering the condition (6) we get inequality (15). The estimation (15) indicates that, the function $g(x, \eta; \mu)$ is the element of the space X_2 .

Consider the operator T on the banach spaces X_1 and X_2 , which is defined by equation

$$\begin{aligned} TA(x, \eta) &= \int_0^{\infty} g(x, \xi; \mu) [Q(\xi) - Q(x)] A(\xi, \eta) d\xi - \\ &- \int_0^{\infty} \left[\frac{1}{2} P^{-\frac{1}{2}}(x) \omega \exp(-|x - \xi| \omega) P^{-\frac{1}{2}}(x) (P(\xi) - P(x)) + g_{\xi}^1(x, \eta; \mu) P(\xi) \right] A(\xi, \eta) d\xi \end{aligned}$$

It is easy to shown that if the operator functions $P(x)$ and $Q(x)$ satisfies the conditions 1)-6), then for sufficiently large μ the operator T is contractive on the spaces X_1 and X_2 and on the banach spaces $X_3^{(p)}$ ($p \geq 1$), $X_1^{(s)}$, $X_2^{(s)}$, $X_4^{(s)}$ ($s \geq 0$), which elements are operator functions $A(x, s)$ on H ($0 < x, s < \infty$).

Consider again integral equation (12), which we write as follows

$$G(x, \eta; \mu) = g(x, \eta; \mu) - TG(\xi, \eta; \mu) \quad (17)$$

Since for sufficiently large μ the operator T is contractive on all enumerated banach spaces, so the equation (17) has a unique solution, which could be obtained by iteration, if the operator function $g(x, \eta; \mu)$ belongs to the corresponding space.

From the estimation (14) follows that $g(x, \eta; \mu) \in X_3^{(1)}$. Therefore the function $G(x, \eta; \mu)$ for sufficiently large μ belongs to the space $X_3^{(1)}$, if the operator functions

$P(x)$ and $Q(x)$ satisfies the conditions 1)-5). If in addition to these conditions the condition 6) is valid, then $g(x, \eta; \mu) \in X_2$ and so $G(x, \eta; \mu)$ is the element of the space X_2 for sufficiently large μ .

By imposing some additional conditions (see [4]) on the operator functions $P(x)$ and $Q(x)$, we can show belonging of the operator function $g(x, \eta; \mu)$ to the space $X_4^{(\pm\frac{1}{2})}$ and therefore belonging of $G(x, \eta; \mu)$ to these spaces.

Following the papers [1], [2] we can show that the operator function $G(x, \eta; \mu)$ is strongly continuous in variables (x, η) , there exists a strong derivative $\frac{\partial G}{\partial \eta}$ and the equation (3) is true. According the method of [1] can be also proved the validity of equations (4), (5).

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