Investigation of the Resolvent of Equation of Second Order with Normal Operator Coefficients on the Semi-Axis

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1. INTRODUCTION

Consider differential operator $L$ in the space $H_1 = L_2([0, \infty), H)$, where $H$ is a separable Hilbert space generated with the expression

$$l(y) = -(P(x)y')' + Q(x)y \quad (0 \leq x < \infty)$$

(1)

and boundary conditions

$$y'(0) - hy(0) = 0$$

(2)

here $y \in H_1$, derivatives accepted as strong, for each fixed $x$ $P(x)$ is a uniformly differentiable bounded operator on $H$, generally speaking, $h$ is a unbounded self-adjoint operator on $H$ and $h \cdot P^{-\frac{1}{2}}(x) = P^{-\frac{1}{2}}(x) \cdot h$.

In this paper Green’s function of the problems (1) - (2) is investigated. Note that, in $P(x) \equiv E$ and $h = 0$, Green’s function for self adjoint Sturm Liouville equation was first studied [1] and in [8], respectively, in the case when $Q(x)$ for each $x$ is a normal
operator on $H$ was investigated in [2], [3]. Green’s function and asymptotic behavior of the eigenvalues of the problems (1)-(2) in the self-adjoint case was studied [4] and [7]. The essential difference of this work from previous work [8] is the presence in the boundary condition an unbounded operator $h$, and $Q(x)$ is a normal operator.

Let’s specify basic boundary conditions for operator functions $P(x), Q(x)$ and operator $h$, in which we’ll investigate Green’s function of the problems (1)-(2).

1) The operator function $P(x)$ is uniformly everywhere differentiable for all $f \in H$

$$m(f, f) \leq (P(x)f, f) \leq M \cdot (f, f), M > 0$$

2) The operators $Q(x)$ are normal and lower bounded for all $x \in [0, \infty)$ on $H$. The operator $Q(x)$ for all $x \in [0, \infty)$ is the inverse of completely continuous operator and its eigenvalues are laid on complex plane out of sector $\Lambda_0 = \{ \lambda : |\arg \lambda - \pi| < \varepsilon_0 \}$, were $\varepsilon_0$ is constant and $0 < \varepsilon_0 < \pi$.

3) There exist the positive constants $A$ and $0 < a < 3/2$ such that, for all $x \in [0, \infty)$ and $|\xi - x| \leq 1$ the following inequality is true

$$\|Q^a(x)Q(\xi) - Q(x)\| < A |x - \xi|$$

4) There exist constant $B$ such that, for $|\xi - x| > 1$ and for all $x \in [0, \infty)$ the inequality

$$\left\|K(\xi) \exp \left\{ -\frac{1}{2} |x - \xi| \cdot \omega \right\} \right\|_H < B$$

is fulfilled, where

$$K(x) = P^{-\frac{1}{2}}(x)Q(x)P^{-\frac{1}{2}}, \quad \omega = \left\{ K(x) + \mu P^{-1}(x) \right\}^{1/2}.$$ 

5) For all $x$ and $\xi$ from $[0, \infty)$ the inequalities

$$\left\|Q(x)P^{\pm \frac{1}{2}}(x)Q^{-1}(x)\right\| < C, \quad \left\|Q(\xi)P^{-\frac{1}{2}}(x)P^{\frac{1}{2}}(\xi)Q^{-1}(\xi)\right\| < C$$

$$\|\omega + h\|^{-1} < C, \quad \|\omega + h\| < C, \quad \|e^{-\omega x}(\omega + h)\| < C$$

are fulfilled.

6) Since $Q(x)$ for all $x$ is the inverse of completely continuous operator, then $K(x) = P^{-\frac{1}{2}}(x)P^{-\frac{1}{2}}$ for all $x$ is also inverse of completely continuous operator. Let denote by $|\beta_1(x)| \leq |\beta_2(x)| \leq ... \leq |\beta_n(x)| \leq ...$ the absolute values of eigenvalues of the operator $K(x)$, which we’ll assume as measurable function. Then suppose that for all $x$ series $\sum_{i=1}^{\infty} |\beta_i(x)|^{-3/2}$ is convergent and its sum $F(x)$ belong to class $L_1 [0, \infty)$. 
2. MAIN RESULTS

The following theorem is the main result of this work.

**Theorem.** If the basic boundary conditions 1)-6) are fulfilled, then for sufficiently large \( \mu > 0 \) there exist an inverse operator \( R_\mu = (L + \mu E)^{-1} \) being an integral operator the operator kernel \( G(x, \eta; \mu) \) which will be called the Green (operator) function of the operator \( L \). \( G(x, \eta; \mu) \) is an operator on \( H \) that depends on two variables \( x \) and \( \eta \) \( (0 \leq x, \eta < \infty) \) and parameter \( \mu \) and satisfies the conditions:

a) \( G(x, \eta; \mu) \) is strongly continuous in variables \( (x, \eta) \);

b) there exists a strong derivative \( \frac{\partial G}{\partial \eta} \), though

\[
\frac{\partial G}{\partial \eta}(x, x + 0; \mu) - \frac{\partial G}{\partial \eta}(x, x - 0; \mu) = -P^{-1}(x)
\]  

(3)

c) \( -\left( G^1_\eta \cdot P(\eta) \right)^1 \eta + G \{Q(\eta) + |\mu| E\} = 0, \ (\eta \neq x) \)  

(4)

\[
\left( \frac{\partial G}{\partial \eta}(x; \eta; \mu) - G(x; \eta; \mu) h \right)_{\eta=0} = 0
\]  

(5)

**Proof.** For proving of this theorem we refer to the paper [1]. Now deduce the following banach spaces of operator-valued functions \( A(x, \eta) \) on \( H \) \( (0 \leq x, \eta < \infty) \), \( X_1, X_2, X_3^{(p)} \) \( (p \geq 1) \) \( X_1^{(s)}, X_2(s), X_4^{(s)}, (s > 0) \), \( X_5 \) where norms correspondingly defined as follows

\[
\|A(x, \eta)\|_{X_1}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)\|_H^2 \, d\eta \right\},
\]

\[
\|A(x, \eta)\|_{X_2}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)\|_2^2 \, d\eta \right\},
\]

Here \( \|A(x, \eta)\|_2 \) is Hilbert-Schmidt (absolute norm) norm of operator function \( A(x, \eta) \) on \( H \).

\[
\|A(x, \eta)\|_{X_3^{(p)}}^2 = \left\{ \sup_{0<x<\infty} \int_0^\infty \|A(x, \eta)\|_H^p \, d\eta \right\}^{1/p}
\]

\[
\|A(x, \eta)\|_{X_1^{(s)}}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)Q^s(\eta)\|_H^2 \, d\eta \right\}
\]

\[
\|A(x, \eta)\|_{X_2^{(s)}}^2 = \int_0^\infty dx \left\{ \int_0^\infty \|A(x, \eta)Q^s(\eta)\|_2^2 \, d\eta \right\}
\]
\[ \|A(x, \eta)\|_{X_4}^2 = \sup_{0 < x < \infty} \int_{-\infty}^{\infty} \|A(x, \eta)Q^\alpha(\eta)\|_H d\eta \]

\[ \|A(x, \eta)\|_{X_5}^2 = \sup_{0 < x < \infty} \sup_{0 < \eta < \infty} \|A(x, \eta)\|_H \]

Completeness of these spaces in case of \( x, \eta \in (-\infty, \infty) \) was established in [1].

Let \( \omega = K(x) + \mu P^{-1}(x) > 0 \), where \( K(x) = P^{-\frac{1}{2}}(x)Q(x)P^{-\frac{1}{2}}(x) \) \( X(\eta) = \chi \omega \eta + \omega^{-1} \cdot \text{sh} \omega \eta \).

Consider the function

\[ g(x, \eta; \mu) = \begin{cases} P^{-\frac{1}{2}}(x)X(\eta)(\omega + h)^{-1}e^{-\omega x}P^{-\frac{1}{2}}(x), & \eta \leq x \\ P^{-\frac{1}{2}}(x)e^{-\omega x}(\omega + h)^{-1}X(x)P^{-\frac{1}{2}}(x), & \eta \leq x \end{cases} \] (6)

It is easy check that the function \( g(x, \eta; \mu) \) satisfies the equation

\[ P(x)\frac{\partial^2 g}{\partial \eta^2} = \{Q(x) + \mu\} g(x, \eta; \mu) \] (7)

and the conditions

\[ \left( \frac{\partial g}{\partial \eta} - h g \right)_{\eta=0} = 0 \] (8)

\[ \frac{\partial g}{\partial \eta} \bigg|_{\eta=x+0} - \frac{\partial g}{\partial \eta} \bigg|_{\eta=x-0} = -P^{-1}(x) \] (9)

By using equation (6) the function \( g(x, \eta; \mu) \) could be represented as follows

\[ g(x, \eta; \mu) = \begin{cases} \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1}(1 + h\omega^{-1})e^{-(x-\eta)\omega}P^{-\frac{1}{2}}(x) \{1 + O(\|\omega^{-1}\|)\}, & \eta \leq x \\ \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1}(1 + h\omega^{-1})e^{-(x-\eta)\omega}P^{-\frac{1}{2}}(x) \{1 + O(\|\omega^{-1}\|)\}, & \eta > x \end{cases} \] (10)

Since \( \|\omega^{-1}\| = 0(1) \) uniformly on \( x \in (0, \infty) \) if \( \mu \to +\infty \), so it follows from (1) that

\[ g(x, \eta; \mu) = \frac{1}{2}P^{-\frac{1}{2}}(x)\omega^{-1}, e^{-|x-\eta|\omega}, P^{-\frac{1}{2}}(x) \{1 + 0(1)\} \] (11)

Consider the integral equation

\[ G(x, \eta; \mu) = g(x, \eta; \mu) - \int_0^\infty g(x, \xi; \mu) \{Q(\xi) - Q(x)\} G(\xi, \eta; \mu) d\xi + \int_0^\infty \left[ \frac{1}{2}P^{-\frac{1}{2}}(x)\omega \exp(-|x-\xi|\omega)P^{-\frac{1}{2}}(x) (P(\xi) - P(x)) + g^1_\xi(\xi, \eta; \mu) P^1(\xi)G(\xi, \eta; \mu) \right] d\xi \] (12)
It will be shown below that for sufficiently large \( \mu > 0 \) the equation (12) could be solved by iteration method and this solution is the Green’s operator function for operator \( L \).

First, let’s estimate norm of operator function \( g(x, \eta; \mu) \). For all \( \lambda \) laid on complex plane out of sector \( \Lambda_0 \) the following inequality is true

\[
|\lambda + \mu| \geq \mu \sin \varepsilon_0
\]

Since

\[
\left\{ K(x) + \mu P^{-1}(x) \right\} f, f \right) = \left\{ Q(x) + \mu E \right\} P^{-1/2}(x)f, P^{-\frac{1}{2}}(x)f \right) \geq \gamma(f, f) = (\mu - d)M^{-1}
\]

\[
\left\{ \omega^{-1} \exp(-|x-\eta|\omega) \right\} f, f \right) \leq \gamma^{-\frac{1}{2}} \exp \left( -|x-\eta| \frac{\gamma^{\frac{1}{2}}}{\gamma^2} \right)
\]

(13)

so from spectral expansion of operator \( g(x, \eta; \mu) \) and inequality (13) follows that

\[
\|g(x, \eta; \mu)\|_H \leq \frac{1 + 0(1)}{2m \cdot \delta_0^{1/2} \cdot \gamma^{1/2}} \exp \left( -|x-\eta| \delta_0^{1/2} \cdot \gamma^{1/2} \right)
\]

where \( \delta_0 = \sin \varepsilon_0 \).

Here and later we’ll select the square root of \( (\lambda + \mu) \) such that \( \text{Re}(\lambda + \mu)^{1/2} > 0 \).

That is why

\[
\int_0^\infty \|g(x, \eta; \mu)\|_H^2 d\eta \leq \frac{(1 + 0(1))^2}{4m^2 \cdot \delta_0^{3/2} \cdot \gamma^{3/2}}
\]

(14)

From this estimation follows that \( g(x, \eta; \mu) \in X_3^{(2)} \)

Denote by \( \|A\|_2 \) the Hilbert-Schmidt norm of the operator \( A \) on the space \( H \).

\[
\int_0^\infty \left\{ \int_0^\infty \|g(x, \eta; \mu)\|_H^2 d\eta \right\} dx \leq C \int_0^\infty \sum_{i=1}^\infty (\beta_i(x) + \mu M^{-1})^{-3/2} < \infty
\]

(15)

For this purpose, note that for all \( f \in D\{K\} \) the inequality

\[
\left\{ K(x) + \mu M^{-1} \right\} f, f \right) \leq \omega^2 f, f \right) \leq \left\{ K(x) + \mu m^{-1} \right\} f, f \right)
\]

is fulfilled, in addition there exist the operators \( \{K(x) + \mu M^{-1}\}^{-1}, \omega^2, \{K(x) + \mu m^{-1}\}^{-1} \) for all \( x \) and they are completely continuous. Therefore for all \( g \in H \) the following inequality is true

\[
\left\{ K(x) + \mu^{-1} \right\} g, g \right) \leq \omega^{-2} g, g \right) \leq \left\{ K(x) + \mu M^{-1} \right\} g, g \right)
\]
This implies that (see [5])

\[ \lambda_j^{-2}(x) \leq (\beta_j(x) + \mu M^{-1})^{-1} \]  \hspace{1cm} (16)

where \( \lambda_j(x) \) eigenvalues of operator \( \omega \).

Then we have:

\[ \|g(x, \eta; \mu)\|_2^2 \leq \frac{1}{4} \left\| P^{-\frac{1}{2}}(x) \right\| \| \omega^{-1} \cdot \exp (-|x - \eta| \omega)\|_2^2 \cdot \left\| P^{-\frac{1}{2}}(x) \right\| (1 + 0(1)) \]

Form inequality (14) follows that

\[ \|\omega^{-1} \cdot \exp (-|x - \eta| \omega)\|_2^2 = \sum_{j=1}^{\infty} \lambda_j^{-2}(x) \exp (-2|x - \eta| \lambda_j(x)) \leq \]

\[ \sum_{j=1}^{\infty} (B_j(x) + \mu M^{-1})^{-1} \exp \left\{ -2|x - \eta| (B_j(x) + \mu M^{-1})^{1/2} \right\} \]

Therefore we have

\[ \|g(x, \eta; \mu)\|_2 \leq \frac{(1 + 0(1))}{4m^2} \leq \sum_{j=1}^{\infty} (B_j(x) + \mu M^{-1})^{-1} \exp \left\{ |x - \eta| (B_j(x) + \mu M^{-1})^{1/2} \right\} \]

Hence by considering the condition (6) we get inequality (15). The estimation (15) indicates that, the function \( g(x, \eta; \mu) \) is the element of the space \( X_2 \).

Consider the operator \( T \) on the banach spaces \( X_1 \) and \( X_2 \), which is defined by equation

\[ TA(x, \eta) = \int_0^\infty g(x, \xi; \mu) [Q(\xi) - Q(x)] A(\xi, \eta) d\xi - \]

\[ - \int_0^\infty \left[ \frac{1}{2} P^{-\frac{1}{2}}(x) \omega \exp (-|x - \xi| \omega) P^{-\frac{1}{2}}(x) (P(\xi) - P(x)) + g^1(x, \eta; \mu) P(\xi) \right] A(\xi, \eta) d\xi \]

It is easy to shown that if the operator functions \( P(x) \) and \( Q(x) \) satisfies the conditions 1)-6), then for sufficiently large \( \mu \) the operator \( T \) is contractive on the spaces \( X_1 \) and \( X_2 \) and on the banach spaces \( X_3^{(p)} (p \geq 1), X_1^{(s)}, X_2^{(s)}, X_3^{(s)} \) \( (s \geq 0) \), which elements are operator functions \( A(x, s) \) on \( H \ (0 < x, s < \infty) \).

Consider again integral equation (12), which we write as follows

\[ G(x, \eta; \mu) = g(x, \eta; \mu) - TG(\xi, \eta; \mu) \]  \hspace{1cm} (17)

Since for sufficiently large \( \mu \) the operator \( T \) is contractive on all enumerated banach spaces, so the equation (17) has a unique solution, which could be obtained by iteration, if the operator function \( g(x, \eta; \mu) \) belongs to the corresponding space.

From the estimation (14) follows that \( g(x, \eta; \mu) \in X_3^{(1)} \). Therefore the function \( G(x, \eta; \mu) \) for sufficiently large \( \mu \) belongs to the space \( X_3^{(1)} \), if the operator functions
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$P(x)$ and $Q(x)$ satisfies the conditions 1)-5). If in addition to these conditions the condition 6) is valid, then $g(x, \eta; \mu) \in X_2$ and so $G(x, \eta; \mu)$ is the element of the space $X_2$ for sufficiently large $\mu$.

By imposing some additional conditions (see [4]) on the operator functions $P(x)$ and $Q(x)$, we can show belonging of the operator function $g(x, \eta; \mu)$ to the space $X_4^{\pm \frac{1}{2}}$ and therefore belonging of $G(x, \eta; \mu)$ to these spaces.

Following the papers [1], [2] we can show that the operator function $G(x, \eta; \mu)$ is strongly continuous in variables $(x, \eta)$, there exists a strong derivative $\frac{\partial G}{\partial \eta}$ and the equation (3) is true. According the method of [1] can be also proved the validity of equations (4), (5).

References


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