Harnack Inequality and Continuity of Solutions for Quasilinear Elliptic Equations in Sobolev Spaces with Variable Exponent

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Abstract

We investigate regularity properties of solutions to the quasilinear elliptic equations in Sobolev spaces with variable exponent, we prove the Harnack's inequality and continuity of solutions.

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1 Introduction

In the present paper we study the regularity of boundary points for solutions to the quasilinear elliptic equations:

$$-\operatorname{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = 0, \quad (1)$$

Our purpose is to establish the Harnack's inequality

$$\operatorname{ess\,sup}_{B(x,R)} u \leq C(\operatorname{ess\,inf}_{B(x,R)} u + R) \quad (2)$$

where C is independent of u and the ball $B = B(x_0, R)$ for $x_0 \in \Omega$, $R > 0$ and we prove the continuity of solutions for (1).

Harnack's inequality and other regularity results for (1) require additional assumptions on the function $p(\cdot)$; see the counterexamples in [6]. The so called logarithmic Hölder continuity condition seems to be the right one for our purposes. This condition was originally introduced by Zhikov [15] in the context of the Lavrentiev phenomenon for solutions of (1), and it has turned out to be a useful tool in regularity and other applications, see, e.g., [1, 2].

For the existence and uniqueness of solutions $u \in \mathcal{W}^{1,p(x)}(\Omega)$ where $1 < p(x) < d$ for all $x \in \Omega$, of the variational Dirichlet problem associated with the quasilinear elliptic equation (1) see [4], these solutions are obtained by the $p(\cdot)$ -obstacle problem.

A typical example for the operator $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{A}(x, \nabla u) = |\nabla u|^{p(x)-2} \nabla u$ and $\mathcal{B}(x, u) = |u|^{p(x)-2} u$ respectively, for all $x \in \mathbb{R}^d$ thus $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$.

Our problem has been studied in many paper see e.g. [11, 12]. Olli Toivanen [11] proved that this problem has a solution when the operator $\mathcal{B}$ depends on x , u , ∇u where $\delta(x) = p(x) - 1$. The main aim of this section is to generalize the condition on $\delta(x)$, We are interested in the case that the operator $\mathcal{B}$ depends only on x and u , satisfying the previous hypothesis **(H3)**, where $p(x) - 1 \leq \delta(x) < p^*(x)$, knowing that the study of the case where δ satisfies the condition $0 \leq \delta(x) < p(x) - 1$ is already investigated in several articles. The contribution of this paper is to verify the Harnack principle for a weak solutions of quasilinear elliptic equations (1) by using the trick of modified test functions under our assumptions **(H3)** below.

In the first section, we introduce some generalization and position of the problem. In second section we give some basic facts about variable exponent spaces and a rough overview of properties of solutions of the prototype equality. In section 3, we generalize, with detailed proofs, Harnack's inequality (2) to all quasilinear elliptic equations (1) with growth conditions of a non-standard form. In last section, we present the concluding remarks.

2 Some preliminaries

We start this section with some definitions and main results of Lebesgue spaces with variable exponent, and Sobolev spaces modeled upon them. For each open bounded subset Ω of $\mathbb{R}^d$ ($d \geq 2$), we define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $p : \Omega \rightarrow$

$]1, +\infty[$ called a **variable exponent** and we denote $p^- := \operatorname{ess\,inf}_{x \in \overline{\Omega}} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x \in \overline{\Omega}} p(x)$.

We introduce also **the convex modular**

$$\varrho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If the exponent is bounded, i.e., if $p^+ < \infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called **the Luxemburg norm**.

One central property of $L^{p(\cdot)}(\Omega)$ is that the norm and the modular topologies coincide, i.e., $\varrho_{p(\cdot)}(u_n) \rightarrow 0$ if and only if $\|u_n\|_{p(\cdot)} \rightarrow 0$.

We denote by $L^{p'(\cdot)}(\Omega)$ **the conjugate space** of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 1 (Generalized Hölder inequality [14]) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

We define the variable exponent Sobolev space (see [9], [5], [8], [14]) by

$$\mathcal{W}^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \|\nabla u\| \in L^{p(\cdot)}(\Omega)\}.$$

with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \quad \forall u \in \mathcal{W}^{1,p(\cdot)}(\Omega).$$

The local Sobolev space $\mathcal{W}_{\text{loc}}^{1,p(\cdot)}(\Omega)$ consists of functions u that belong to $\mathcal{W}_{\text{loc}}^{1,p(\cdot)}(U)$ for all open sets U compactly contained in Ω . The Sobolev space with zero boundary values, $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$, is defined as the completion of $C_0^\infty(\Omega)$ in the norm of $\mathcal{W}^{1,p(\cdot)}(\Omega)$.

Let $p^*(x)$ be **the Sobolev conjugate exponent** of $p(x)$ defined by

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{for } p(x) < d, \\ +\infty & \text{for } p(x) \geq d. \end{cases}$$

We assume further on that, there exist positive constant C such that the function p satisfies **logarithmic Hölder continuity condition** if :

$$(\star) \begin{cases} \exists C > 0 : |p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \text{ for } |x-y| < \frac{1}{2}, \\ 1 < p^- \leq p^+ < d. \end{cases}$$

Proposition 2 (The $p(\cdot)$ -Poincaré inequality)

Let Ω be a bounded open set and let $p : \Omega \rightarrow [1, \infty[$ satisfy $(\star)$. There exists a constant C , depending only on $p(\cdot)$ and Ω , such that the inequality

$$\|u\|_{p(\cdot)} \leq C \|\nabla u\|_{p(\cdot)} \quad \forall u \in \mathcal{W}_0^{1,p(\cdot)}(\Omega).$$

Lemma 2.1 (Sobolev inequality [7]) Let Ω be a bounded open set and u in $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$. There exists a constant C such that

$$\left(\int_{\Omega} |u|^{\frac{dp(x)}{d-1}} dx \right)^{\frac{d-1}{dp(x)}} \leq C \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{p(x)}} \quad (3)$$

Proposition 3

Assuming $p^- > 1$, the spaces $\mathcal{W}^{1,p(\cdot)}(\Omega)$ and $\mathcal{W}_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

Throughout the paper we suppose that the functions $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Carathéodory function satisfying the following assumptions:

$$(H1) \quad |\mathcal{A}(x, \xi)| \leq \beta[k(x) + |\xi|^{p(x)-1}];$$

$$(H2) \quad \mathcal{A}(x, \xi)\xi \geq \nu|\xi|^{p(x)};$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^d$, where $k(x)$ is a positive bounded function lying in $L^{p'(x)}(\Omega)$ and $\beta, \nu > 0$.

In this paper we suppose that the function $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is given Carathéodory function and the following condition is satisfied:

$$(H3) \quad |\mathcal{B}(x, \zeta)| \leq g(x) + |\zeta|^{\delta(x)};$$

for a.e. $x \in \Omega$, all $\zeta \in \mathbb{R}^d$, where g is a positive bounded function lying in $L^{p'(x)}(\Omega)$ and $p(x) - 1 \leq \delta(x) < p^*(x)$.

Remark 2.1 Under the assumption $(\star)$ Harnack's inequality and local Hölder continuity follow from Moser or De Giorgi-type procedure; see [10, 2, 3]. An interesting feature of this theory is that estimates are intrinsic in the sense that they depend on the solution itself. For example, supersolutions are assumed to be locally bounded and Harnack-type estimates in [2] depend on this bound.

Definition 2.1 We say that a $u \in \mathcal{W}_{loc}^{1,p(\cdot)}(\Omega)$ is a $p(\cdot)$ -solution of (1) in Ω provided that for all $\varphi \in \mathcal{W}_0^{1,p(\cdot)}(\Omega)$ if,

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} \mathcal{B}(x, u) \varphi dx = 0.$$

Definition 2.2 A function $u \in \mathcal{W}_{loc}^{1,p(\cdot)}(\Omega)$ is termed $p(\cdot)$ -supersolutions of (1), if and only if, for all non-negative functions $\varphi \in \mathcal{W}_0^{1,p(\cdot)}(\Omega)$ we have,

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} \mathcal{B}(x, u) \varphi dx \geq 0.$$

A function u is a **$p(\cdot)$ -subsolution** in Ω if $-u$ is a $p(\cdot)$ -supersolution in Ω , and a solution in Ω if it is both a super- and a $p(\cdot)$ -subsolution in Ω .

3 Harnack inequality and continuity of solutions to quasilinear elliptic equations

The Harnack inequality is a very important estimate in the study of $p(\cdot)$ -solutions of quasilinear elliptic equations.

3.1 Main result

We start by adapting a standard Caccioppoli type estimate for $p(\cdot)$ -supersolution of (1). Then we use the Caccioppoli estimate to show that for a fixed, non-negative $p(\cdot)$ -supersolution u , the inequality (2).

The Harnack inequality is indispensable as a tool in the qualitative theory of second-order elliptic equations. In particular, it implies continuity of weak solutions see [11, 13].

By non-linearity we mean that if $p \neq 2$ then the weak solutions do not form a linear space. However the set of weak solutions is closed under constant multiplication. By celebrated De Giorgi's method and Moser's iteration the weak solutions are locally Hölder continuous and satisfy Harnack's inequality.

Remark 3.1 Our notation is rather standard. Various constants are denoted by C and the value of the constant may differ even on the same line. The quantities on which the constants depend are given in the statements of the theorems and lemmas.

Lemma 3.1 Let E be a measurable subset of $\mathbb{R}^d$. For all nonnegative measurable functions ψ and φ defined on E ,

$$\int_E \psi \varphi^{p^-} dx \leq \int_E \psi dx + \int_E \psi \varphi^{p(x)} dx$$

The following Caccioppoli estimate is the key result of this paper; and it is a modification of [[7], Lemma 3.2]. The new feature in the estimate is the choice of a test function which includes the variable exponent.

Lemma 3.2 (*Caccioppoli estimate*)

Suppose that u is a nonnegative $p(\cdot)$ -supersolution in B_{4R} . Let E be a measurable subset of B_{4R} and $\eta \in \mathcal{C}_0^\infty(B_{4R})$ such that $0 \leq \eta \leq 1$. Then for every $\gamma_0 < 0$ there is a constant C depending on p and γ_0 such that the inequality:

$$\int_E \eta^{p_{B_{4R}}^+} u^{\gamma-1} |\nabla u|^{p^-} dx \leq C \int_{B_{4R}} (u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} + \eta^{p_{B_{4R}}^+} u^{\gamma-1}) \quad (4)$$

$$+ (u^{\gamma+p(x)-1} dx + \eta^{p_{B_{4R}}^+} u^{\gamma+\delta(x)}) dx$$

holds for every $\gamma < \gamma_0 < 0$ and $p(x) - 1 \leq \delta(x) < p^*(x)$.

Proof 1 Let $\theta = p_{B_{4R}}^+$. We want to test with the function $\varphi = \eta^\theta u^\gamma$.

To this end we show that $\varphi \in \mathcal{W}_0^{1,p(\cdot)}(B_{4R})$. Since η has a compact support in B_{4R} , it is enough to show that $\varphi \in \mathcal{W}^{1,p(\cdot)}(\Omega)$. We observe that $\varphi \in L^{p(x)}(\Omega)$ since $|u^\gamma| \eta^\theta \leq R^\gamma$. Furthermore, we have

$$\nabla \varphi = \gamma \eta^\theta u^{\gamma-1} \nabla u + \theta \eta^{\theta-1} u^\gamma \nabla \eta$$

Using the fact that u is a $p(\cdot)$ -supersolution and φ is a nonnegative test function we find that

$$0 \leq \int_{B_{4R}} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_{B_{4R}} \mathcal{B}(x, u) \varphi dx$$

$$= \int_{B_{4R}} \gamma \eta^\theta u^{\gamma-1} \mathcal{A}(x, \nabla u) \cdot \nabla u dx + \int_{B_{4R}} \theta \eta^{\theta-1} u^\gamma \mathcal{A}(x, \nabla u) \cdot \nabla \eta dx + \int_{B_{4R}} \mathcal{B}(x, u) \eta^\theta u^\gamma dx$$

We denote the left-hand side of the next inequality by I . Since γ is a negative number this implies by the structural conditions (H1), (H2) and (H3) that

$$I = |\gamma_0| \nu \int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx$$

$$\leq \beta \int_{B_{4R}} \theta \eta^{\theta-1} u^\gamma \left(k(x) + |\nabla u|^{p(x)-1} \right) |\nabla \eta| dx + \int_{B_{4R}} \left(g(x) + u^{\delta(x)} \right) \eta^\theta u^\gamma dx$$

$$\leq \underbrace{\beta \theta \int_{B_{4R}} \eta^{\theta-1} u^\gamma |\nabla u|^{p(x)-1} |\nabla \eta| dx}_{=I_1} + \underbrace{\beta \theta \int_{B_{4R}} \eta^{\theta-1} u^\gamma k(x) |\nabla \eta| dx}_{=I_2} + \underbrace{\int_{B_{4R}} (g(x) + u^{\delta(x)}) \eta^\theta u^\gamma dx}_{=I_3}$$

Using **Young's inequality**, $0 < \varepsilon \leq 1$, we obtain the first estimate

$$I_1 \leq \int_{B_{4R}} \left(\frac{1}{\varepsilon} \right)^{p(x)-1} \left(\eta^{\theta - \frac{\theta}{p'(x)} - 1} u^{\frac{\gamma+p(x)-1}{p(x)}} |\nabla \eta| \right)^{p(x)} + \varepsilon \left(\eta^{\frac{\theta}{p'(x)}} u^{\gamma - \frac{\gamma+p(x)-1}{p(x)}} |\nabla u|^{p(x)-1} \right)^{p'(x)} dx$$

$$\leq \left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} \eta^{\theta-p(x)} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx$$

$$\leq \left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx$$

Next we estimate the last two integrals I_2 and I_3 .

To estimate the integral I_2 , we denote $0 \leq v = \eta + |\nabla \eta|$ and k is a positive bounded function there exist a constant $M > 0$, and by Young's inequality we have

$$\begin{aligned} I_2 &= \int_{B_{4R}} \eta^{\theta-1} u^\gamma k(x) |\nabla \eta| dx \leq M \int_{B_{4R}} v \eta^{\theta-1} u^\gamma dx \\ &\leq M \left(\left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} \eta^{\theta-p(x)} u^{\gamma+p(x)-1} v^{p(x)} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \right) \\ &\leq M \left(\left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} v^{p(x)} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \right) \\ &\leq M \left(\left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} (1 + |\nabla \eta|)^{p(x)} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \right) \end{aligned}$$

Using the inequality $(\varphi + \psi)^{p(\cdot)} \leq 2^{p(\cdot)-1}(|\varphi|^{p(\cdot)} + |\psi|^{p(\cdot)})$, for $p(\cdot) \geq 1$,

$$\begin{aligned} I_2 &\leq \left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} 2^{p(x)-1} (1 + |\nabla \eta|^{p(x)}) dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \\ &\leq \frac{2^{\theta-1}}{\varepsilon^{\theta-1}} \int_{B_{4R}} u^{\gamma+p(x)-1} (1 + |\nabla \eta|^{p(x)}) dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \\ &\leq \left(\frac{2}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx + \left(\frac{2}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx \end{aligned}$$

To estimate the integral I_3 , and by the assumption g is a positive bounded function there exist a $N > 0$ such that

$$\begin{aligned} I_3 &= \int_{B_{4R}} (g(x) + u^{\delta(x)}) \eta^\theta u^\gamma dx \leq \int_{B_{4R}} (N + u^{\delta(x)}) \eta^\theta u^\gamma dx \\ &\leq \int_{B_{4R}} \eta^{\theta-1} u^{\gamma+\delta(x)} dx + N \overbrace{\int_{B_{4R}} \eta^{\theta-1} u^\gamma dx}^{=I'_3} \end{aligned}$$

By Young's inequality we have

$$I'_3 \leq \left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx$$

Thus

$$I_3 \leq \int_{B_{4R}} \eta^{\theta-1} u^{\gamma+\delta(x)} dx + \left(\frac{1}{\varepsilon} \right)^{\theta-1} \int_{B_{4R}} u^{\gamma+p(x)-1} dx + \varepsilon \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx$$

Therefore

$$(|\gamma_0|\nu - \theta\beta\varepsilon) \int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx \leq \theta\beta \frac{(2^{\theta-1} + 1)}{\varepsilon^{\theta-1}} \int_{B_{4R}} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx \\ + \varepsilon(\theta\beta + 1) \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx + \frac{2^{\theta-1}\theta\beta + 1}{\varepsilon^{\theta-1}} \int_{B_{4R}} u^{\gamma+p(x)-1} dx + \int_{B_{4R}} \eta^{\theta-1} u^{\gamma+\delta(x)} dx$$

By choosing $\varepsilon = \min\{1, \frac{|\gamma_0|\nu}{2\theta\beta}\}$

$$\int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx \leq \\ \leq C_1 \int_{B_{4R}} u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx + C_2 \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx + C_3 \int_{B_{4R}} u^{\gamma+p(x)-1} dx + \\ + C_4 \int_{B_{4R}} \eta^{\theta-1} u^{\gamma+\delta(x)} dx$$

Where $C_1 = (2^{\theta-1} + 1)(\frac{2\theta\beta}{|\gamma_0|\nu})$, $C_2 = 1 + \frac{1}{\theta\beta}$, $C_3 = (2^{\theta-1} + \frac{1}{\theta\beta})(\frac{2\theta\beta}{|\gamma_0|\nu})^\theta$, and $C_4 = \frac{2}{|\gamma_0|\nu}$.

We take $C = C_i$ for $i = 1, 2, 3, 4$. and we have

$$\int_{B_{4R}} \eta^{p_{B_{4R}}^+} u^{\gamma-1} |\nabla u|^{p(x)} dx \leq C \int_{B_{4R}} (u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} + \eta^{p_{B_{4R}}^+} u^{\gamma-1}) \\ + (u^{\gamma+p(x)-1} dx + \eta^{p_{B_{4R}}^+ - 1} u^{\gamma+\delta(x)}) dx$$

By the lemma 3.1 we obtain

$$\int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p^-} dx \leq \int_{B_{4R}} \eta^\theta u^{\gamma-1} dx + \int_{B_{4R}} \eta^\theta u^{\gamma-1} |\nabla u|^{p(x)} dx$$

and using the previous inequality we obtain the claim.

$$\int_E \eta^{p_{B_{4R}}^+} u^{\gamma-1} |\nabla u|^{p^-} dx \leq C \int_{B_{4R}} (u^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} + \eta^{p_{B_{4R}}^+} u^{\gamma-1}) \\ + (u^{\gamma+p(x)-1} dx + \eta^{p_{B_{4R}}^+ - 1} u^{\gamma+\delta(x)}) dx$$

So the proof of lemma is achieved.

3.2 Weak Harnack Inequality

In this section we prove a weak Harnack inequality for $p(\cdot)$ -supersolutions to (1).

Throughout this subsection we write $v = u + R$ where u is a nonnegative $p(\cdot)$ -supersolution and $0 \leq R \leq 1$.

We start by the following technical lemma that is need later. These results are mainly from [7]

Lemma 3.3 *If the exponent $p(\cdot)$ is log-Hölder continuous, then $R^{-p(x)} \leq CR^{-p_E}$ where $x \in E \subset B_R$ and $R > 0$.*

Lemma 3.4 *Let f be a positive measurable function and assume that the exponent $p(\cdot)$ is log-Hölder continuous. Then*

$$\int_{B_r} f^{p_{B_{4R}}^+ - p_{B_{4R}}^-} dx \leq C \|f\|_{L^s(B_r)}^{p_{B_{4R}}^+ - p_{B_{4R}}^-}$$

for any $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$

Now we have everything ready for the iteration. We write

$$\Phi(f, q, B_r) = \left(\int_{B_r} f^q dx \right)^{\frac{1}{q}}$$

for a nonnegative measurable function f .

The point is that the Moser iteration technique used in [7] remains valid under our consideration.

Lemma 3.5 *Let u is a nonnegative $p(\cdot)$ -supersolution of (1) in B_{4R} and let $R \leq \rho < r \leq 3R$. Then the inequality*

$$\Phi(v, q\tau, B_r) \leq C^{\frac{1}{|\tau|}} (1 + |\tau|)^{\frac{p_{B_{4R}}^+}{|\tau|}} \left(\frac{r}{r - \rho} \right)^{\frac{p_{B_{4R}}^+}{|\tau|}} \Phi(v, \tau \frac{d}{d-1}, B_\rho) \quad (5)$$

holds for every $\tau < 0$ and $1 < q < \frac{d}{d-1}$. The constant C depends on d, p , and the $L^{q^s}(B_{4R})$ -norm of u with $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$ and all structure constants and functions of (H1), (H2) and (H3) hypothesis.

Proof 2 Let $\theta^+ = p_{B_{4R}}^+$ and $\theta^- = p_{B_{4R}}^-$, we take $\gamma = \tau - \theta^- + 1$. In (4) of the lemma 3.2 we have

$$\begin{aligned} & \int_{B_{4R}} \eta^{\theta^+} v^{\tau - \theta^-} |\nabla u|^{\theta^-} dx \\ & \leq C \int_{B_{4R}} \overbrace{(\eta^{\theta^+} u^{\tau - \theta^-} + u^{\tau - \theta^- + p(x)} |\nabla \eta|^{p(x)})}^{=I_1} + \overbrace{(u^{\tau - \theta^- + p(x)} + \eta^{\theta^+ - 1} u^{\tau - \theta^- + 1 + \delta(x)})}^{=I_2} dx \end{aligned} \quad (6)$$

Now we take the test function $\eta \in \mathcal{C}_0^\infty(B_r)$ with $0 \leq \eta \leq 1$, $\eta = 1$ in B_ρ , and

$$|\nabla \eta| \leq \frac{Cr}{R(r - \rho)}$$

Next we went to estimate the integral I_2 by the integral

$$\left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}} \quad (7)$$

Using lemma 3.3,3.4, the first integral I_1 is estimated by (7) see [7]. Finally, for the second integral I_2 we have by Hölder's inequality

$$\begin{aligned} \int_{B_r} u^{\tau-\theta^-+p(x)} dx &\leq \int_{B_r} v^{\tau-\theta^-+p(x)} dx \\ &\leq C \left(\int_{B_r} v^{q'(p(x)-\theta^-)} dx \right)^{\frac{1}{q'}} \left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}} \\ &\leq C \left(1 + \|v\|_{L^{q's}(B_{4R})}^{q'(\theta^+-\theta^-)} \right)^{\frac{1}{q'}} \left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}} \end{aligned}$$

On the other hand and since $p(x) - 1 \leq \delta(x) < p^*(x)$

$$\begin{aligned} \int_{B_r} \eta^{\theta^+-1} u^{\tau-\theta^-+1+\delta(x)} dx &\leq \int_{B_r} v^{\tau-\theta^-+1+\delta(x)} dx \\ &\leq C \left(\int_{B_r} v^{q'(\delta(x)-\theta^-+1)} dx \right)^{\frac{1}{q'}} \left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}} \\ &\leq C \left(1 + \|v\|_{L^{q's}(B_{4R})}^{q'(\theta^+-\theta^-)} \right)^{\frac{1}{q'}} \left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}} \end{aligned}$$

Therefore, the second integral I_2 is estimated by (3).

In lemma 2.1 we take $u = v^{\frac{\tau}{\theta^-}} \eta^{\frac{\theta^-}{\theta^-}}$ and we use the inequality:

$$(a + b)^{p(\cdot)} \leq 2^{p(\cdot)-1} (|a|^{p(\cdot)} + |b|^{p(\cdot)}), \quad p(\cdot) \geq 1,$$

We obtain

$$\begin{aligned} \left(\int_{B_\rho} v^{\frac{d\tau}{d-1}} \right)^{\frac{d-1}{d}} &\leq C \left(\int_{B_r} \left(v^{\frac{\tau}{\theta^-}} \eta^{\frac{\theta^-}{\theta^-}} \right)^{\frac{d\theta^-}{d-1}} dx \right)^{\frac{d-1}{d}} \\ &\leq CR^{\theta^-} \int_{B_r} |\nabla (v^{\frac{\tau}{\theta^-}} \eta^{\frac{\theta^-}{\theta^-}})|^{\theta^-} dx \\ &\leq C|\tau|^{\theta^-} \int_{B_r} \eta^{\theta^+} v^{\tau-\theta^-} |\nabla u|^{\theta^-} dx + C \int_{B_r} v^\tau \eta^{\theta^+-\theta^-} |\nabla \eta|^{\theta^-} dx \end{aligned}$$

Using inequality (4) we have

$$\begin{aligned}
\left(\int_{B_\rho} v^{\frac{d\tau}{d-1}} \right)^{\frac{d-1}{d}} &\leq C \left(\int_{B_r} \left(v^{\frac{\tau}{\theta^-}} \eta^{\frac{\theta^-}{\theta^-}} \right)^{\frac{d\theta^-}{d-1}} dx \right)^{\frac{d-1}{d}} \\
&\leq CR^{\theta^-} \int_{B_r} |\nabla(v^{\frac{\tau}{\theta^-}} \eta^{\frac{\theta^-}{\theta^-}})|^{\theta^-} dx \\
&\leq C|\tau|^{\theta^-} \int_{B_r} \left(\eta^{\theta^+} u^{\tau-\theta^-} + u^{\tau-\theta^-+p(x)} |\nabla \eta|^{p(x)} + v^\tau \eta^{\theta^+-\theta^-} |\nabla \eta|^{\theta^-} \right) dx \\
&\quad + C \int_{B_r} \left(u^{\tau-\theta^-+p(x)} + \eta^{\theta^+-1} u^{\tau-\theta^-+1+\delta(x)} \right) dx \\
&\leq C(1+|\tau|)^{\theta^+} \left(1 + \|v\|_{L^{q's}(B_{4R})}^{q'(\theta^+-\theta^-)} \right)^{\frac{1}{q'}} \left(\frac{r}{r-\rho} \right)^{\theta^+} \left(\int_{B_r} v^{q\tau} dx \right)^{\frac{1}{q}}
\end{aligned}$$

Finally, since $\tau < 0$ we have

$$\Phi(v, q\tau, B_r) \leq C^{\frac{1}{|\tau|}} (1+|\tau|)^{\frac{p_{B_{4R}}^+}{-|\tau|}} \left(\frac{r}{r-\rho} \right)^{\frac{p_{B_{4R}}^+}{-|\tau|}} \Phi(v, \tau \frac{d}{d-1}, B_\rho)$$

So the proof of lemma is achieved.

The proofs of the following results can be found in [11]. and [7], respectively

Lemma 3.6 *Assume that u is a nonnegative $p(\cdot)$ -supersolution of (1) in B_{4R} and $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$. Then there exist constants $q_0 > 0$ and C depending on d, p , and $L^{q's}(B_{4R})$ -norm of u such that:*

$$\Phi(v, q_0, B_{3R}) \leq C \Phi(v, -q_0, B_{3R}) \quad (8)$$

Theorem 3.7 (Weak Harnack inequality) *Let u be a non-negative $p(\cdot)$ -super-solution of (1) in B_{4R} and $1 < q < \frac{d}{d-1}$. Then*

$$\left(\int_{B_{2R}} u^{q_0} dx \right)^{\frac{1}{q_0}} \leq C \left(\operatorname{ess\,inf}_{x \in B_R} u(x) + R \right) \quad (9)$$

Where q_0 is the exponent from Lemma 3.5 and C depends on d, p, q and the $L^{q's}(B_{4R})$ -norm of u with $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$ and all structure constants and functions of (H1), (H2) and (H3) hypothesis.

Theorem 3.8 *Let u be a non-negative $p(\cdot)$ -subsolution of (1) in B_{4R} and $1 < q < \frac{d}{d-1}$. Then*

$$\operatorname{ess\,sup}_{x \in B_R} u(x) \leq C \left(\int_{B_{2R}} u^t dx \right)^{\frac{1}{t}} + R \quad (10)$$

Where $t > 0$ and C depends on d, p, q and the $L^{q's}(B_{4R})$ -norm of u with $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$.

To combine (9) and (10) we obtain the crucial theorem.

Theorem 3.9 (Harnack's inequality) *Let u be a non-negative $p(\cdot)$ -solution of (1) in Ω and let $B_{4R} \subset \Omega$ and $1 < q < \frac{d}{d-1}$, and $s > p_{B_{4R}}^+ - p_{B_{4R}}^-$ with $0 \leq R \leq 1$. Then*

$$\operatorname{ess\,sup}_{x \in B_R} u(x) \leq C \left(\operatorname{ess\,inf}_{x \in B_R} u(x) + R \right)$$

Where C depends on d, p, q and the $L^{q^s}(B_{4R})$ -norm of u with and all structure constants and functions of (H1), (H2) and (H3) hypothesis.

4 Conclusion

We showed that $p(\cdot)$ -solutions are locally bounded, locally bounded $p(\cdot)$ -supersolutions satisfy the weak Harnack inequality and locally bounded $p(\cdot)$ -solutions satisfy Harnack's inequality. In the proof of Harnack's inequality the Caccioppoli estimate is used for the function $u + R$, where R is a radius of a ball.

References

- [1] E. Acerbi, G. Mingione, Regularity results for a class of functionals with nonstandard growth, Arch. Ration. Mech. Anal. 156 (2) (2001) 121-140.
- [2] Y. A. Alkhutov, The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition, Differ. Uravn. 33 (12) (1997) 1651-1660, 1726.
- [3] Y. A. Alkhutov and O. V. Krashenninnikova, Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition, English translation in Izvestiya: Mathematics 68 (2004), no. 6, 1063-1117.
- [4] A. Baalal & A. Qabil, The $p(\cdot)$ -obstacle problem for quasilinear elliptic equations in sobolev spaces with variable exponent, International Journal of Applied Mathematics and Statistics, Vol. 48; Issue 18, 2013.
- [5] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Academic Press, New York, 2011.
- [6] P. Hasto, Counter examples of regularity in variable exponent Sobolev spaces, Contemp. Math, 367 (2005) 133-143.

- [7] P. Harjulehto, Juha Kinnunen, T. Lukkari, Unbounded Supersolutions of Nonlinear Equations with Nonstandard Growth, Boundary Value Problems, Volume 2007, Article ID 48348, 20 pages doi:10.1155/2007/48348.
- [8] X. L. Fan and D. Zhao, On the generalised Orlicz-Sobolev Space $\mathcal{W}^{k,p(x)}(\Omega)$, J. Gansu Educ. College 12(1)(1998) 1-6.
- [9] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41(116)(1991), 592-618.
- [10] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577-591.
- [11] Olli. Toivanen, Harnack's inequality for general solutions with nonstandard growth, Ann. Acad. Sci. Fenn. Math., 37 (2012), 571-577.
- [12] Yongqiang Fu, Yingying Shan, Removability of singularity for nonlinear elliptic equations with $p(x)$ -growth, Electronic Journal of Qualitative Theory of Differential Equations 2013, No. 58, 1-17.
- [13] X. Zhang and X. Liu, The local boundedness and Harnack inequality of $p(x)$ -Laplace equation. J. Math. Anal. Appl., 332:209-218, 2007.
- [14] D. Zhao, W. J. Qiang and X. L. Fan, On generalized Orlicz spaces $L^{p(x)}(\Omega)$, J. Gansu Sci. 9(2) 1997 1-7.
- [15] V. V. Zhikov, On Lavrentiev's phenomenon, Russian J. Math. Phys, 3(2) (1995) 249-269.

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