Asymptotics of the Solution of a Boundary Value Problem for One-Characteristic Differential Equation Degenerating into a Parabolic Equation in an Infinite Strip

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Abstract

In an infinite strip a boundary value problem for a third order non-classic type equation degenerating into a second order parabolic equation is considered. The total asymptotic expansion in small parameter of the solution of the problem under consideration is constructed and the remainder term is estimated.

Keywords: Asymptotics, Boundary layer function, Remainder term

1. Introduction

In studying some real phenomena with non-uniform transitions from one physical characteristics to other ones, we have to investigate singularly perturbed boundary value problems. A lot of mathematicians were interested in such problems. But non-classical singular perturbed equations in comparison with classic equations were studied not enough.

M. V. Vishik and L. A. Lusternik in [5] introduced so-called one – characteristic equations. They called the equations of odd order $2k + 1$ and of the form

$$L_{2k+1} \equiv A_1(A_{2k}u) + B_{2k}u = f \quad (1)$$

one-characteristic ones if A_1 is an operator of first order, A_{2k} is an elliptic operator of at most $2k$ order. Obviously, only the characteristics of the first order operator A_1 will be real characteristics of equation (1). In the same paper they investigated mutual degeneration of elliptic and one-characteristic equations in finite domains, and constructed only the first terms of the asymptotics of solution of the considered problems.

It should be noted that in references, mainly asymptotics of the solution of boundary value problems for different equations was studied only in finite domains. In the papers [1], [2], for one-characteristic equations that were studied in [5] the boundary value problems were considered in infinite domains and the total asymptotics of solution of these problems were constructed.

In the present paper, in an infinite strip $P = \{(t, x) | 0 \leq t \leq 1, -\infty < x < +\infty\}$ we consider the following boundary value problem:

$$L_\varepsilon u \equiv \varepsilon^2 \frac{\partial}{\partial t} (\Delta u) - \varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au = f(t, x), \quad (2)$$

$$u|_{t=0} = 0, \quad u|_{t=1} = 0, \quad \frac{\partial u}{\partial t} \Big|_{t=1} = 0, \quad (3)$$

$$\lim_{|x| \rightarrow +\infty} u = 0, \quad (4)$$

where $\varepsilon > 0$ is a small parameter, $\Delta \equiv \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$, $a > 0$ is a constant, $f(t, x)$ is a given smooth function. In [3] the boundary value problem for equation (2) in a rectangle is considered and total asymptotics of the solution of considered problem is constructed.

Our goal is to construct the asymptotic expansion in small parameter of the solution of boundary value problem (2)-(4). In constructing the asymptotics we are guided by the M.I. Vishik-L.A. Lusternik method and conduct iterative processes.

2. The first iterative process

In the first iterative process, we'll look for the approximate solution of equation (2) in the form

$$W = W_0 + \varepsilon W_1 + \dots + \varepsilon^n W_n, \quad (5)$$

and the functions $W_i(t, x); i = 0, 1, \dots, n$ will be chosen so that

$$L_\varepsilon W = O(\varepsilon^{n+1}). \quad (6)$$

Substituting (5) in (2), and equating the terms with the same powers of ε , for determining $W_i; i = 0, 1, \dots, n$ we get the following recurrently connected equations:

$$\frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t, x), \quad (7)$$

$$\frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = \frac{\partial^2 W_0}{\partial t^2}, \quad (8)$$

$$\frac{\partial W_k}{\partial t} - \frac{\partial^2 W_k}{\partial x^2} + aW_k = \frac{\partial^2 W_{k-1}}{\partial t^2} - \frac{\partial}{\partial t}(\Delta W_{k-2}); k = 2, 3, \dots, n. \quad (9)$$

Equations (7), (8), (9) differ only by the right sides. Equation (7) is obtained from equation (2) for $\varepsilon = 0$, and is called a degenerate equation corresponding to equation (2).

Obviously, it is impossible to use all boundary conditions of (3) for equations (7), (8), (9). For the equations (7), (8), (9) with respect to t it should be used the first condition from (3), and with respect to x the both conditions from (4), i.e.

$$W_i|_{t=0} = 0, \quad (-\infty < x < +\infty); \quad (10)$$

$$\lim_{|x| \rightarrow +\infty} W_i = 0, (0 \leq t \leq 1); i = 0, 1, \dots, n. \quad (11)$$

Problem (7), (10), (11) for $i = 0$ is said to be a degenerate problem corresponding to problem (2)-(4).

The following lemma is valid

Lemma 1. Let $f(t, x)$ be a function given in P , having continuous derivatives with respect to t up to the $(n+3)$ -th order inclusively, be infinitely differentiable with respect to x and satisfy the condition

$$\sup_x (1 + |x|^l) \left| \frac{\partial^k f(t, x)}{\partial t^{k_1} \partial x^{k_2}} \right| = C_{l_{k_1 k_2}}^{(1)} < +\infty, \quad (12)$$

where l is a nonnegative number, $k = k_1 + k_2$, $k_1 \leq n+3$, k_2 is arbitrary, $C_{l_{k_1 k_2}}^{(1)} > 0$.

Then the function $W_0(t, x)$ being the solution of problem (7), (10), (11) for $i = 0$, in P has continuous derivatives with respect to t up to the $(n+4)$ -th order inclusively, is infinitely differentiable with respect to x and satisfies the condition

$$\sup_x (1 + |x|^l) \left| \frac{\partial^k W_0(t, x)}{\partial t^{k_1} \partial x^{k_2}} \right| = C_{l_{k_1 k_2}}^{(2)} < +\infty, \quad (13)$$

where $k_1 \leq n+4$, $C_{l_{k_1 k_2}}^{(2)} > 0$.

The proof of this lemma is cited in the paper [4].

The remaining functions $W_1, W_2, \dots, W_n$, in expansion (5) will be determined sequentially from boundary value problems (8), (9), (10), (11) for $i = 1, 2, \dots, n$. From lemma 1 it follows that the functions W_i being the solutions of problem (8), (9), (10), (11) for $i = 1, 2, \dots, n$ will have continuous derivatives with respect to x up to the $(n+3-i)$ -th order, and condition (13) for the function W_i will be satisfied for $k_1 \leq n+3-i$; $i = 1, 2, \dots, n$.

From (5) and (10), (11) we get that the constructed function W satisfies the following boundary conditions:

$$W|_{t=0} = 0, (-\infty < x < +\infty); \lim_{|x| \rightarrow +\infty} W = 0, (0 \leq t \leq 1). \quad (14)$$

The function W doesn't satisfy, generally speaking, boundary conditions for $t=1$ from (3). Therefore, conducting the second iterative process we should construct a boundary layer-function V near the boundary $t=1$ so that the sum $W+V$ could satisfy the boundary conditions

$$(W+V)|_{t=1} = 0, \quad \frac{\partial}{\partial t}(W+V)|_{t=1} = 0. \quad (15)$$

3. The second iterative process-construction of boundary layer functions

The first iterative process is conducted on the base of decomposition (2) of the operator L_ε . For conducting other iterative process by means of which a boundary layer function will be constructed near the boundary $t=1$, at first it is necessary to write a new decomposition of the operator L_ε near this boundary. For that we make change of variables: $1-t = \varepsilon\tau$, $x = x$. The new decomposition of the operator L_ε in the coordinates (τ, x) has the form

$$L_{\varepsilon,1} \equiv \varepsilon^{-1} \left[- \left(\frac{\partial^3}{\partial \tau^3} + \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau} \right) + \varepsilon \left(- \frac{\partial^2}{\partial x^2} + a \right) - \varepsilon^2 \frac{\partial^3}{\partial \tau \partial x^2} \right]. \quad (16)$$

We look for a boundary layer function V near the boundary $t=1$ in the form

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \dots + \varepsilon^{n+1} V_{n+1}, \quad (17)$$

as the approximate solution of the equation

$$L_{\varepsilon,1} V = 0. \quad (18)$$

Substituting the expression for V from (17) in (18), taking into account (16) and making comparison of the terms at the same powers of ε , for determining the function $V_j(\tau, x); j=0,1,\dots,n+1$ we get the following recurrently connected equations:

$$\frac{\partial^3 V_0}{\partial \tau^3} + \frac{\partial^2 V_0}{\partial \tau^2} + \frac{\partial V_0}{\partial \tau} = 0, \quad (19)$$

$$\frac{\partial^3 V_1}{\partial \tau^3} + \frac{\partial^2 V_1}{\partial \tau^2} + \frac{\partial V_1}{\partial \tau} = - \frac{\partial^2 V_0}{\partial x^2} + a V_0, \quad (20)$$

$$\frac{\partial^3 V_k}{\partial \tau^3} + \frac{\partial^2 V_k}{\partial \tau^2} + \frac{\partial V_k}{\partial \tau} = - \frac{\partial^2 V_{k-1}}{\partial x^2} + a V_{k-1} - \frac{\partial^3 V_{k-2}}{\partial \tau \partial x^2}; k = 2, 3, \dots, n+1. \quad (21)$$

For finding boundary conditions for equations (19), (20), (21) it is necessary to substitute expansions (5), (17) for W and V to (15) and make comparison of the terms at the same powers of ε . Then we get:

$$V_i|_{\tau=0} = -W_i|_{t=1}; i = 0, 1, \dots, n; V_{n+1}|_{\tau=0} = 0, \quad (22)$$

$$\frac{\partial V_0}{\partial \tau}\bigg|_{\tau=0} = 0, \frac{\partial V_j}{\partial \tau}\bigg|_{\tau=0} = \frac{\partial W_{j-1}}{\partial t}\bigg|_{t=1}; j = 1, 2, \dots, n+1. \quad (23)$$

Now construct the functions $V_0, V_1, \dots, V_{n+1}$. From (19), (22) for $i = 0$ and (23) we have that the function V_0 is a boundary layer type solution of equation (19), satisfying the boundary conditions

$$V_0|_{\tau=0} = -W_0|_{t=T}, \frac{\partial V_0}{\partial \tau}\bigg|_{\tau=0} = 0. \quad (24)$$

The characteristic equation corresponding to ordinary differential equation (19) in addition to the zero root has two non-zero roots $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ with negative real parts. This fact provides regularity of degeneration of problem (2)-(4) on the boundary $t = 1$.

The boundary layer type solution of problem (19), (24) has the form

$$V_0(\tau, x) = \frac{W_0(T, x)}{\lambda_1 - \lambda_2} (\lambda_2 e^{\lambda_1 \tau} - \lambda_1 e^{\lambda_2 \tau}). \quad (25)$$

By lemma 1, from (25) it follows that the function $V_0(\tau, x)$ and all its even derivatives with respect to x vanish as $|x| \rightarrow +\infty$.

Knowing the function V_0 , we determine the function V_1 as a boundary layer solution of equation (20), satisfying the boundary conditions:

$$V_1|_{\tau=0} = -W_1|_{t=T}, \frac{\partial V_1}{\partial \tau}\bigg|_{\tau=0} = \frac{\partial W_0}{\partial t}\bigg|_{t=T}. \quad (26)$$

From (25) it follows that the right side of equation (20) has the form

$$f_1 = m_1(x) e^{\lambda_1 \tau} + m_2(x) e^{\lambda_2 \tau}, \quad (27)$$

where $m_1(x)$, $m_2(x)$ are determined by the following equalities

$$\begin{aligned} m_1(x) &= \frac{\lambda_2}{\lambda_1 - \lambda_2} \left[a W_0(T, x) - \frac{\partial^2 W_0(T, x)}{\partial x^2} \right], \\ m_2(x) &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \left[\frac{\partial^2 W_0(T, x)}{\partial x^2} - a W_0(T, x) \right]. \end{aligned} \quad (28)$$

Following (27) we get that equation (19) has a particular solution in the form

$$V_1^{(1)} = [c_{10}(x) + c_{11}(x)\tau]e^{\lambda_1\tau} + [c_{20}(x) + c_{21}(x)\tau]e^{\lambda_2\tau}, \quad (29)$$

and the functions $c_{10}(x)$, $c_{11}(x)$, $c_{20}(x)$, $c_{21}(x)$ are expressed by the functions $W_0(T, x)$ and $\frac{\partial^2 W_0(T, x)}{\partial x^2}$. They may be determined by the method of undetermined coefficients.

Represent V_1 in the form $V_1 = V_1^{(1)} + V_1^{(2)}$. Then $V_1^{(2)}$ will be the boundary layer type solution of the following problem:

$$\frac{\partial^3 V_1^{(2)}}{\partial \tau^3} + \frac{\partial^2 V_1^{(2)}}{\partial \tau^2} + \frac{\partial V_1^{(2)}}{\partial \tau} = 0; \quad V_1^{(2)} \Big|_{\tau=0} = \varphi_1(x), \quad \frac{\partial V_1^{(2)}}{\partial \tau} \Big|_{\tau=0} = \psi_1(x), \quad (30)$$

where

$$\begin{aligned} \varphi_1(x) &= -W_1(T, x) - c_{10}(x), \\ \psi_1(x) &= \frac{\partial W_0(T, x)}{\partial t} - c_{11}(x) - \lambda_1 c_{10}(x) - c_{21}(x) - \lambda_2 c_{20}(x). \end{aligned} \quad (31)$$

Obviously, the boundary layer type solution of problem (30) has the form

$$V_1^{(2)}(\tau, x) = \frac{1}{\lambda_1 - \lambda_2} \left\{ [\psi_1(x) - \lambda_2 \varphi_1(x)] e^{\lambda_1 \tau} + [\lambda_1 \varphi_1(x) - \psi_1(x)] e^{\lambda_2 \tau} \right\}. \quad (32)$$

From (29) and (32) comes out that the function V_1 is the sum of $V_1^{(1)}$ and $V_1^{(2)}$ and is determined by the formula

$$V_1(\tau, x) = [a_{10}(x) + a_{11}(x)\tau]e^{\lambda_1\tau} + [b_{10}(x) + b_{11}(x)\tau]e^{\lambda_2\tau}, \quad (33)$$

by $a_{10}(x)$, $a_{11}(x)$, $a_{20}(x)$, $a_{21}(x)$ the functions

$$\begin{aligned} a_{10}(x) &= \frac{1}{\lambda_1 - \lambda_2} [\psi_1(x) - \lambda_2 \varphi_1(x)] + c_{10}(x), \quad a_{11}(x) = c_{11}(x), \\ b_{10}(x) &= \frac{1}{\lambda_1 - \lambda_2} [\lambda_1 \varphi_1(x) - \psi_1(x)] + c_{20}(x), \quad b_{11}(x) = c_{21}(x) \end{aligned} \quad (34)$$

are denoted.

According to (31), (33), (34) we have that the function $V_1(\tau, x)$ and all its even derivatives with respect to x vanish as $|x| \rightarrow +\infty$.

By constructing the functions $V_k; k = 2, 3, \dots, n+1$ we use the following statement.

Lemma 2. The functions V_k being the boundary layer type solutions of equations (21) for $k = 2, 3, \dots, n+1$ and satisfying the corresponding conditions from (22), (23) are determined by the formula

$$V_k(\tau, x) = \left[\sum_{i=0}^k a_{ki}(x) \right] e^{\lambda_i \tau} + \left[\sum_{i=0}^k b_{ki}(x) \right] e^{\lambda_{i+1} \tau}; k = 2, 3, \dots, n+1. \quad (35)$$

The coefficients $a_{ki}(x)$, $b_{ki}(x)$ are expressed uniformly by the functions $W_0(T, x)$, $W_1(T, x)$, ..., $W_k(T, x)$ and their derivatives of first order with respect to t , and with respect to x only of even order.

Proof. The lemma is proved by the mathematical induction method. It was shown above that the functions V_0, V_1 are determined by formula (35). Now assume that the statement of the lemma is valid for $V_0, V_1, V_2, \dots, V_{k-1}$, and prove that it is valid for V_k as well. Note that the functions V_{k-1}, V_{k-2} enter into the right side of equation (21) for V_k and by assumption these functions are determined by formula (35).

Repeating the reasoning conducted in determining the function V_1 , we can affirm that V_k is also determined by formula (35).

Lemma 2 is proved.

Multiply all the functions V_j by the smoothing function and the obtained new functions again denote by V_j ; $j = 0, 1, \dots, n+1$. As all the functions $V_j(\tau, x)$; $j = 0, 1, \dots, n+1$ vanish as $|x| \rightarrow +\infty$, then from (14) and (17) it follows that the sum $\tilde{U} = W + V$ constructed by us, in addition to (15) satisfies also the boundary conditions

$$(W + V)|_{t=0} = 0, \quad \lim_{|x| \rightarrow +\infty} (W + V) = 0. \quad (36)$$

Denote the difference of the exact solution of problem (2)-(4) and $\tilde{U}$ by

$$U - \tilde{U} = \varepsilon^{n+1} z, \quad (37)$$

and call $\varepsilon^{n+1} z$ a remainder term. Now we should estimate the remainder term.

4. Estimation of remainder term

It holds the following statement.

Lemma 3. For the function z it is valid the estimation

$$\varepsilon^2 \left\| \frac{\partial z}{\partial t} \right\|_{L_2(-\infty, +\infty)}^2 + \varepsilon \left\| \frac{\partial z}{\partial t} \right\|_{L_2(P)}^2 + \left\| \frac{\partial z}{\partial x} \right\|_{L_2(P)}^2 + c_1 \|z\|_{L_2(P)}^2 \leq c_2, \quad (38)$$

where $c_1 > 0, c_2 > 0$ are the constants independent of ε .

Proof. Acting on both sides of equality (37) by the appropriate decompositions of the operator L_ε , and taking into account equations (2), (7)-(9) and (19)-(21), we have

$$L_\varepsilon z = F(\varepsilon, t, x), \quad (39)$$

where $F(\varepsilon, t, x) = F_1(\varepsilon, t, x) + F_2(\varepsilon, \tau, x)$ is a uniformly bounded function. The function $F_1(\varepsilon, t, x)$ has the form

$$F_1 = \frac{\partial^2 W_n}{\partial t^2} - \frac{\partial}{\partial t}(\Delta W_{n-2}) - \varepsilon \frac{\partial}{\partial t}(\Delta W_{n-1}),$$

and $F_2(\varepsilon, \tau, x)$ near the boundary $t = T$ has the form

$$F_2 = \frac{\partial^2 V_n}{\partial x^2} - aV_n + \varepsilon \left(\frac{\partial V_{n+1}}{\partial x^2} - aV_{n+1} \right) + \frac{\partial^3 V_{n-1}}{\partial \tau \partial x^2} + \varepsilon \frac{\partial^3 V_n}{\partial \tau \partial x^2} + \varepsilon^2 \frac{\partial^3 V_{n+1}}{\partial \tau \partial x^2}.$$

Obviously, z will satisfy the boundary conditions

$$z|_{t=0} = z|_{t=T} = 0, \quad \frac{\partial z}{\partial t} \Big|_{t=T} = 0, \quad z|_{x=0} = z|_{x=1} = 0. \quad (40)$$

Multiplying the both sides of equation (39) scalarly by z , and integrating the left side of the obtained equality with regard to boundary conditions (40), after some transformations we get estimation (38).

Lemma 3 is proved.

5. Conclusion

The obtained results may be generalized in the form of the following statement.

Theorem. Let $f(t, x)$ be a given function in P , having continuous derivatives with respect to t to the $(n+3)$ -th order inclusively, with respect to x be infinitely differentiable, and satisfy condition (12). Then for the solution of problem (2)-(4) it holds the asymptotic representation

$$u = \sum_{i=0}^n \varepsilon^i W_i + \sum_{j=0}^{n+1} \varepsilon^j V_j + \varepsilon^{n+1} z,$$

where the functions W_i are determined by the first iterative process, V_j are boundary layer type functions near the boundary $t = 1$ and are determined by the second iterative process, $\varepsilon^{n+1} z$ is a remainder term, and estimation (38) is valid for the function z .

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