Almost Periodic and Almost Automorphic Solutions to Some Classes of Stochastic Neutral Evolution Equations

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Abstract

We investigate some classes of stochastic neutral evolution equations in a real separable Hilbert space. Sufficient conditions for the existence and uniqueness of square-mean almost periodic and square-mean almost automorphic mild solutions are derived with the help of the Banach fixed point theorem. An example is given to illustrate the theory.

Keywords: Square-mean almost periodic; Square-mean almost automorphic; fixed point; Exponentially stable semigroup

1 Introduction

The concept of an almost periodic function was introduced by Bohr for numerical valued functions and extended by Bochner for functions with values in Polish spaces. In recent years, the existence of almost periodic on different kinds of deterministic differential equations has been

considerably investigated in lots of publications [1–8] because of its significance and applications in physics, mechanics and mathematical biology.

In [9], Bezandry and Diagana introduced a new concept of a square-mean almost periodicity. They established the existence and uniqueness of square-mean almost periodic mild solutions to some stochastic differential equations and some functional integro-differential stochastic evolution equations in [9, 10].

The concept of almost automorphy was first initiated by Bochner in his landmark paper [11]. Since then the theory of almost automorphic functions has found several developments and applications in the theory of abstract differential equations, partial differential equations, functional-differential equations, and integro-differential equations. For more on these and related issues, we refer the reader to [12–16] and the references therein.

In a very recent paper [17], Fu and Liu introduced a new concept of a square-mean almost automorphic stochastic process. They established the existence and uniqueness of square-mean almost automorphic mild solutions to some stochastic differential equations.

Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation has received much attention in recent years. The existence of almost periodic and almost automorphic solutions to deterministic neutral differential equations has been considerably investigated in lots of publications [1, 4, 13]. However, The existence of square-mean almost periodic and square-mean almost automorphic solutions to stochastic neutral differential equations has not been considered. Motivated by the above discussion, We give in this work some sufficient conditions for the existence and uniqueness of square-mean almost periodic and square-mean almost automorphic mild solutions to some classes of stochastic neutral evolution equations.

2 Preliminaries

For details of this section, we refer the reader to Da Prato and Zabczyk [18]. Throughout this paper, H and K will denote real separable Hilbert spaces with respective norms $\|\cdot\|_H$ and $\|\cdot\|_K$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e, it is right continuous and $\mathcal{F}_0$ contains all P -null sets). We denote by $L_2(K; H)$ the space of all Hilbert-Schmidt operators acting between K and H equipped with the Hilbert-Schmidt norm $\|\cdot\|_2$. For a symmetric nonnegative operator $Q \in L_2(K; H)$ with finite trace we assume that $W(t), t \in R$ is a Q -Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ and with values in K .

The collection of all strongly measurable, square-integrable H -valued random variables, denoted by $L^2(P; H)$, is a Banach space equipped with norm $\|X\|_{L^2(P; H)} = \left(E\|X\|_H^2\right)^{\frac{1}{2}}$, where the expectation E is defined by $E[g] = \int_{\Omega} g(\omega) dP(\omega)$.

Let $K_0 = Q^{\frac{1}{2}}K$ and $L_2^0 = L_2(K_0; H)$ with respect to the norm

$$\|\Phi\|_{L_2^0}^2 = \left\| \Phi Q^{\frac{1}{2}} \right\|_2^2 = \text{Tr}(\Phi Q \Phi^*).$$

We are concerned with the existence almost periodic and almost automorphic solutions to the class of stochastic neutral evolution equations of the form

$$d[x(t) + f(t, Bx(t))] = (Ax(t) + g(t, Cx(t)))dt + \sigma(t, Dx(t))dW(t), \quad t \in R, \quad (1)$$

where A is the infinitesimal generator of a uniformly exponentially stable C_0 -semigroup acting on $L^2(P; H)$, B , C and D are densely defined closed linear operators on $L^2(P; H)$. $f, g : R \times L^2(P; H) \rightarrow L^2(P; H)$ and $\sigma : R \times L^2(P; H) \rightarrow L^2(P; L_2^0)$ are jointly continuous functions. One should point out that various stochastic differential equations, stochastic partial differential equations, stochastic integrodifferential equations, and stochastic functional differential equations can be written of the form Eq. (1).

Let $(B, \|\cdot\|_B)$ be a Banach space. Let $CUB(R; L^2(P; B))$ denote the collection of all stochastic processes $X : R \rightarrow L^2(P; B)$, which are continuous and uniformly bounded. It is then easy to check that $CUB(R; L^2(P; B))$ is a Banach space when it is equipped with the norm:

$$\|X\|_{B, \infty} = \sup_{t \in R} \left(E\|X(t)\|_B^2 \right)^{\frac{1}{2}}.$$

Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be Banach spaces. $L(B_1; B_2)$ and $L(B_1)$ stand respectively for the class of bounded linear operator from B_1 to B_2 and the class of bounded linear operator from B_1 to itself.

Definition 2.1. A continuous stochastic process $X : R \rightarrow L^2(P; B)$ is said to be square-mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in R} E\|X(t + \tau) - X(t)\|_B^2 < \varepsilon.$$

The collection of all stochastic processes $X : R \rightarrow L^2(P; B)$ which are square-mean almost periodic is then denoted by $AP(R; L^2(P; B))$.

The next lemma provides some properties of square-mean almost periodic processes.

Lemma 2.1. [9] If X belongs to $AP(R; L^2(P; B))$, then

- (i) the mapping $t \rightarrow E\|X(t)\|^2$ is uniformly continuous;
- (ii) there exists a constant $N > 0$ such that $E\|X(t)\|^2 \leq N$, for all $t \in R$.

Lemma 2.2. [9] $AP(R; L^2(P; B)) \subset CUB(R; L^2(P; B))$ is a closed subspace.

In view of the above, the space $AP(R; L^2(P; B))$ of square-mean almost periodic processes equipped with the norm $\|\cdot\|_{B, \infty}$ is a Banach space.

Definition 2.2. A function $F : R \times L^2(P; B_1) \rightarrow L^2(P; B_2), (t, Y) \rightarrow F(t, Y)$, which is jointly continuous, is said to be square-mean almost periodic in $t \in R$ uniformly in $Y \in K$ where $K \subset L^2(P; B_1)$ is a compact if for any $\varepsilon > 0$, there exists $l(\varepsilon, K) > 0$ such that any interval of length $l(\varepsilon, K)$ contains at least a number τ for which

$$\sup_{t \in R} E \|F(t + \tau, Y) - F(t, Y)\|_{B_2}^2 < \varepsilon$$

for each stochastic process $Y : R \rightarrow K$.

Lemma 2.3. [9] Let $F : R \times L^2(P; B_1) \rightarrow L^2(P; B_2), (t, Y) \rightarrow F(t, Y)$ be square-mean almost periodic in $t \in R$ uniformly in $Y \in K$ where $K \subset L^2(P; B_1)$ is a compact. Suppose that F is Lipschitz in the following sense:

$$E \|F(t, Y) - F(t, Z)\|_{B_2}^2 \leq ME \|Y - Z\|_{B_1}^2$$

for all $Y, Z \in L^2(P; B_1)$ and for each $t \in R$, where $M > 0$. Then for any square-mean almost periodic process $\Phi : R \rightarrow L^2(P; B_1)$, the stochastic process $t \rightarrow F(t, \Phi(t))$ is square-mean almost periodic.

Definition 2.3. A continuous stochastic process $X : R \rightarrow L^2(P; B)$ is said to be square mean almost automorphic if for every sequence of real numbers $\{s'_n\}_{n \in N}$ there exist a subsequence $\{s_n\}_{n \in N}$ and a stochastic process $Y : R \rightarrow L^2(P; B)$ such that

$$\lim_{n \rightarrow \infty} E \|X(t + s_n) - Y(t)\|_B^2 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} E \|Y(t - s_n) - X(t)\|_B^2 = 0$$

hold for each $t \in R$.

The collection of all square-mean almost automorphic stochastic processes $X : R \rightarrow L^2(P; B)$ is denoted by $AA(R; L^2(P; B))$.

Lemma 2.4. [17] $AA(R; L^2(P; B)) \subset CUB(R; L^2(P; B))$ equipped with the norm $\|\cdot\|_{B, \infty}$ is a Banach space.

Definition 2.4. A function $F : R \times L^2(P; B_1) \rightarrow L^2(P; B_2), (t, Y) \rightarrow F(t, Y)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in R$ for each $Y \in L^2(P; B_1)$ if for every sequence of real numbers $\{s'_n\}_{n \in N}$ there exist a subsequence $\{s_n\}_{n \in N}$ and a stochastic process $\tilde{F} : R \times L^2(P; B_1) \rightarrow L^2(P; B_2)$ such that

$$\lim_{n \rightarrow \infty} E \|F(t + s_n, Y) - \tilde{F}(t, Y)\|_{B_2}^2 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} E \|\tilde{F}(t - s_n, Y) - F(t, Y)\|_{B_2}^2 = 0$$

Lemma 2.5. [17] Let $F : R \times L^2(P; B_1) \rightarrow L^2(P; B_2), (t, Y) \rightarrow F(t, Y)$ be square-mean almost automorphic in $t \in R$ uniformly in $Y \in L^2(P; B_1)$. Suppose that F is Lipschitz in the following sense:

$$E \|F(t, Y) - F(t, Z)\|_{B_2}^2 \leq ME \|Y - Z\|_{B_1}^2$$

for all $Y, Z \in L^2(P; B_1)$ and for each $t \in R$, where $M > 0$. Then for any square-mean almost automorphic process $\Phi : R \rightarrow L^2(P; B_1)$, the stochastic process $t \rightarrow F(t, \Phi(t))$ is square mean almost automorphic.

For convenience, we recall from [19] the mild solution to (1) as follows.

Definition 2.5. A stochastic process $x(t) : [\delta, \delta + a) \rightarrow L^2(P; H)$, $a > 0$, is a mild solution of the neutral stochastic system (1) on $[\delta, \delta + a)$, if the function $s \rightarrow AT(t - s)f(s, x(s - r))$ is integrable on $[\delta, t)$ for every $\delta < t < \delta + a$, and

$$\begin{aligned} x(t) = & T(t)[x(\delta) + f(\delta, Bx(\delta))] - f(t, Bx(t)) - \int_{\delta}^t AT(t - s)f(s, Bx(s))ds \\ & + \int_{\delta}^t T(t - s)g(s, Cx(s))ds + \int_{\delta}^t T(t - s)\sigma(s, Dx(s))dW(s), \quad t \in [\delta, \delta + a) \end{aligned} \quad (2)$$

3 Existence results

Throughout the rest of the work, $(G, \|\cdot\|_G)$ denotes an arbitrary real separable Hilbert spaces, which is continuously embedded into the ground real separable Hilbert spaces H .

To prove our results we always assume that the following conditions are satisfied.

(H1) The operator A is the infinitesimal generator of an exponentially stable semigroup $(T(t))_{t \geq 0}$ such that there exist constants $M > 0$ and $\delta > 0$ with

$$\|T(t)\|_{L(L^2(P; G))} \leq Me^{-\delta t}, \quad \forall t \geq 0.$$

Furthermore, the function $s \rightarrow AT(s)$ defined from $(0, \infty)$ into $L(L^2(P; G))$ is strongly (Lebesgue) measurable and there exist a function $\gamma : [0, \infty) \rightarrow [0, \infty)$ and a constant $\omega > 0$ with $\rho := \int_0^\infty e^{-\omega s} \gamma(s) ds < \infty$ such that

$$\|AT(s)\|_{L(L^2(P; G))} \leq e^{-\omega s} \gamma(s), \quad s > 0.$$

(H2) The functions $f, g(t, x) : R \times L^2(P; H) \rightarrow L^2(P; G)$ is a square-mean almost periodic in $t \in R$ uniformly in $x \in \Omega$ ($\Omega \subset L^2(P; H)$ being a compact subspace). Moreover, f, g is

Lipschitz in the following sense: there exist L_f, L_g for which

$$\begin{aligned} E \|f(t, x) - f(t, y)\|_G^2 &\leq L_f E \|x - y\|_H^2, \\ E \|g(t, x) - g(t, y)\|_G^2 &\leq L_g E \|x - y\|_H^2, \end{aligned}$$

for all stochastic processes $x, y \in L^2(P; H)$ and $t \in R$.

(H3) The functions $\sigma(t, x) : R \times L^2(P; H) \rightarrow L^2(P; L_2(K_0; G))$ is a square-mean almost periodic in $t \in R$ uniformly in $x \in \Omega$ ($\Omega \subset L^2(P; H)$ being a compact subspace). Moreover, σ is Lipschitz in the following sense: there exists L_σ for which

$$E \|\sigma(t, x) - \sigma(t, y)\|_{L_2(K_0; G)}^2 \leq L_\sigma E \|x - y\|_H^2$$

for all stochastic processes $x, y \in L^2(P; H)$ and $t \in R$.

(H4) B, C and D belong to $L(L^2(P; G); L^2(P; H))$ with $\max \left(\|B\|_{L(L^2(P; G); L^2(P; H))}, \|C\|_{L(L^2(P; G); L^2(P; H))}, \|D\|_{L(L^2(P; G); L^2(P; H))} \right) = h$.

(H5) The functions $f, g(t, x) : R \times L^2(P; H) \rightarrow L^2(P; G)$ is a square-mean almost automorphic in $t \in R$ uniformly in $x \in L^2(P; H)$. Moreover, f, g is Lipschitz in the following sense: there exist L_f, L_g for which

$$\begin{aligned} E \|f(t, x) - f(t, y)\|_G^2 &\leq L_f E \|x - y\|_H^2, \\ E \|g(t, x) - g(t, y)\|_G^2 &\leq L_g E \|x - y\|_H^2, \end{aligned}$$

for all stochastic processes $x, y \in L^2(P; H)$ and $t \in R$.

(H6) The functions $\sigma(t, x) : R \times L^2(P; H) \rightarrow L^2(P; L_2(K_0; G))$ is a square-mean almost automorphic in $t \in R$ uniformly in $x \in L^2(P; H)$. Moreover, σ is Lipschitz in the following sense: there exists L_σ for which

$$E \|\sigma(t, x) - \sigma(t, y)\|_{L_2(K_0; G)}^2 \leq L_\sigma E \|x - y\|_H^2$$

for all stochastic processes $x, y \in L^2(P; H)$ and $t \in R$.

The proof of our main results require the following technical lemmas.

Lemma 3.1. Suppose that assumptions (H1) – (H2) and (H4) hold. Define the nonlinear operator Λ_1 by: for each $\phi \in AP(R; L^2(P; G))$,

$$(\Lambda_1 \phi)(t) := \int_{-\infty}^t AT(t-s) f(s, B\phi(s)) ds.$$

Then Λ_1 maps $AP(R; L^2(P; G))$ into itself.

Proof. First, let us check that Λ is well defined. Indeed, if $\phi \in AP(R; L^2(P; G))$, then $B\phi(s)$ is in $AP(R; L^2(P; G))$ as $B \in L(L^2(P; G); L^2(P; H))$. And hence, by Lemma (2.3), the function $f(t, B\phi(t)) \in AP(R; L^2(P; G))$ when $\phi \in AP(R; L^2(P; G))$. In particular, $\|f(t, B\phi(t))\|_{G, \infty} < \infty$. So, we have from the above discussion and (H1)

$$\begin{aligned} E \|(\Lambda\phi)(t)\|_G^2 &= E \left\| \int_{-\infty}^t AT(t-s) f(s, B\phi(s)) ds \right\|_G^2 \\ &\leq \left(\int_{-\infty}^t e^{-\omega(t-s)} \gamma(t-s) ds \right) \left(\int_{-\infty}^t e^{-\omega(t-s)} \gamma(t-s) E \|f(s, B\phi(s))\|_G^2 ds \right) \\ &\leq \rho^2 \|f(t, B\phi(t))\|_{G, \infty}^2 < \infty. \end{aligned}$$

Now, Λ_1 is well defined. By Lemma (2.3), the function $f(t, B\phi(t)) \in AP(R; L^2(P; G))$ when $\phi \in AP(R; L^2(P; G))$. Therefore, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least τ for which

$$E \|f(t+\tau, B\phi(t+\tau)) - f(t, B\phi(t))\|_G^2 < \frac{\varepsilon}{\rho^2}$$

for each $t \in R$.

We also have that

$$\begin{aligned} E \|(\Lambda_1\phi)(t+\tau) - (\Lambda_1\phi)(t)\|_G^2 &= E \left\| \int_{-\infty}^t AT(t-s) [f(s+\tau, B\phi(s+\tau)) - f(s, B\phi(s))] ds \right\|_G^2 \\ &\leq \left(\int_{-\infty}^t e^{-w(t-s)} \gamma(t-s) ds \right) \\ &\quad \times \left(\int_{-\infty}^t e^{-w(t-s)} \gamma(t-s) E \|f(s+\tau, B\phi(s+\tau)) - f(s, B\phi(s))\|_G^2 ds \right) \\ &\leq \rho^2 \|f(s+\tau, B\phi(s+\tau)) - f(s, B\phi(s))\|_{G, \infty}^2 \\ &< \varepsilon \end{aligned}$$

In view of the above, $E \|(\Lambda_1\phi)(t+\tau) - (\Lambda_1\phi)(t)\|_G^2 < \varepsilon$ for each $t \in R$, that is, $(\Lambda_1\phi)t$ is square-mean almost periodic. $\square$

Lemma 3.2. Suppose that assumptions (H1) – (H2) and (H4) hold. Define the nonlinear operator Λ_2 by: for each $\phi \in AP(R; L^2(P; G))$,

$$(\Lambda_2\phi)(t) := \int_{-\infty}^t T(t-s) g(s, C\phi(s)) ds.$$

Then Λ_2 maps $AP(R; L^2(P; G))$ into itself.

Proof. The proof is similar to that of Lemma (3.1) with appropriate adaptations. So we omit it. $\square$

Lemma 3.3. Suppose that assumptions (H1), (H3) and (H4) hold. Define the nonlinear operator Λ_3 by: for each $\phi \in AP(R; L^2(P; G))$,

$$(\Lambda_3\phi)(t) := \int_{-\infty}^t T(t-s)\sigma(s, Dx(s))dW(s).$$

Then Λ_3 maps $AP(R; L^2(P; G))$ into itself.

Proof. The proof is similar to that of Theorem 3.2 in [9] with appropriate adaptations. So we omit it. $\square$

Lemma 3.4. Suppose that assumptions (H4) – (H5) hold. Define the nonlinear operator Λ_4 by: for each $\phi \in AA(R; L^2(P; G))$,

$$(\Lambda_4\phi)(t) := \int_{-\infty}^t AT(t-s)f(s, B\phi(s))ds.$$

Then Λ_4 maps $AA(R; L^2(P; G))$ into itself.

Proof. First, let us check that Λ_4 is well defined. Indeed, if $\phi \in AA(R; L^2(P; G))$, then $B\phi(s)$ is in $AA(R; L^2(P; G))$ as $B \in L(L^2(P; G); L^2(P; H))$. And hence, by Lemma (2.5), the function $f(t, B\phi(t)) \in AA(R; L^2(P; G))$ when $\phi \in AA(R; L^2(P; G))$. In particular, $\|f(t, B\phi(t))\|_{G, \infty} < \infty$. So, we have from the above discussion and (H1)

$$\begin{aligned} E\|(\Lambda_4\phi)(t)\|_G^2 &= E\left\|\int_{-\infty}^t AT(t-s)f(s, B\phi(s))ds\right\|_G^2 \\ &\leq \left(\int_{-\infty}^t e^{-\omega(t-s)}\gamma(t-s)ds\right)\left(\int_{-\infty}^t e^{-\omega(t-s)}\gamma(t-s)E\|f(s, B\phi(s))\|_G^2 ds\right) \\ &\leq \rho^2\|f(t, B\phi(t))\|_{G, \infty}^2 < \infty. \end{aligned}$$

Now, Λ_4 is well defined. To prove that $(\Lambda_4\phi) \in AA(R; L^2(P; G))$, one has to prove that

$$t \rightarrow \Lambda_4\phi(t) = \lim_{r \rightarrow -\infty} \int_r^t AT(t-s)\varphi(s)ds$$

is square-mean almost automorphic, where $\varphi(s) = f(s, Bx(s))$ for each $s \in R$.

Now let s'_n be an arbitrary sequence of real numbers, Since φ is square-mean almost automorphic, there exists a subsequence (s_n) of s'_n such that $\lim_{n \rightarrow \infty} E\|\varphi(t+s_n) - h(t)\|_G^2 = 0$ for each $t \in R$ and $\lim_{n \rightarrow \infty} E\|\varphi(t) - h(t-s_n)\|_G^2 = 0$ for each $t \in R$.

Now

$$\begin{aligned} &E\left\|\int_{-\infty}^{t+s_n} AT(t+s_n-s)\varphi(s)ds - \int_{-\infty}^t AT(t-s)h(s)ds\right\|^2 \\ &= E\left\|\int_{-\infty}^t AT(t-s)\varphi(s+s_n)ds - \int_{-\infty}^t AT(t-s)h(s)ds\right\|^2 \\ &\leq E\left\|\int_{-\infty}^t AT(t-s)[\varphi(s+s_n) - h(s)]ds\right\|^2 \\ &\leq \rho \int_{-\infty}^t e^{-\omega(t-s)}\gamma(t-s)E\|\varphi(s+s_n) - h(s)\|^2 ds. \end{aligned}$$

So, by Lebesgues dominated convergence theorem, $E \left\| \int_{-\infty}^{t+s_n} AT(t+s_n-s) \varphi(s) ds - \int_{-\infty}^t AT(t-s) \varphi(s) ds \right\|^2 = 0$ as $n \rightarrow \infty$ for each $t \in R$.

Similarly, it can be shown that $E \left\| \int_{-\infty}^t AT(t-s) \varphi(s) ds - \int_{-\infty}^{t-s_n} AT(t-s_n-t-s) h(s) ds \right\|^2 = 0$ as $n \rightarrow \infty$ for each $t \in R$.

This shows that $\Lambda_4 \phi(t)$ is a square-mean almost automorphic function. $\square$

Lemma 3.5. Suppose that assumptions (H1) and (H4) – (H5) hold. Define the nonlinear operator Λ_5 by: for each $\phi \in AA(R; L^2(P; G))$,

$$(\Lambda_5 \phi)(t) := \int_{-\infty}^t T(t-s) g(s, C\phi(s)) ds.$$

Then Λ_5 maps $AA(R; L^2(P; G))$ into itself.

Proof. The proof is similar to that of Lemma 3.4 with appropriate adaptations. So we omit it. $\square$

Lemma 3.6. Suppose that assumptions (H1), (H4) and (H6) hold. Define the nonlinear operator Λ_6 by: for each $\phi \in AA(R; L^2(P; G))$,

$$(\Lambda_6 \phi)(t) := \int_{-\infty}^t T(t-s) \sigma(s, Dx(s)) dW(s).$$

Then Λ_6 maps $AA(R; L^2(P; G))$ into itself.

Proof. The proof is similar to that of Theorem 3.1 in [20] with appropriate adaptations. So we omit it. $\square$

Theorem 3.1. Assume the conditions (H1) – (H4) hold. Then (1) has a unique square-mean almost period mild solution on R provided that

$$4 \left[h^2 L_f + \rho^2 R^2 L_f + h^2 \frac{1}{\delta^2} M^2 L_g + h^2 \frac{1}{\delta^2} M^2 L_\sigma \right] < 1. \quad (3)$$

Proof. Let $\Gamma: AP(R; L^2(P; G)) \rightarrow AP(R; L^2(P; G))$ be the map defined by

$$\begin{aligned} \Gamma x(t) = & -f(t, Bx(t)) - \int_{-\infty}^t AT(t-s) f(s, Bx(s)) ds \\ & + \int_{-\infty}^t T(t-s) g(s, Cx(s)) ds + \int_{-\infty}^t T(t-s) \sigma(s, Dx(s)) dW(s), \quad t \in R. \end{aligned} \quad (4)$$

From Lemma 2.3, Lemma 3.1, Lemma 3.2 and Lemma 3.3, it follows that the nonlinear operator Γ is well defined, that is, it maps $AP(R; L^2(P; G)) \rightarrow AP(R; L^2(P; G))$. In the following, we prove that Γ is a contraction.

Clearly,

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\|_G &\leq \|f(t, Bx(t)) - f(t, By(t))\|_G + \left\| \int_{-\infty}^t AT(t-s) [f(s, Bx(s)) - f(s, By(s))] ds \right\|_G \\ &\quad + \left\| \int_{-\infty}^t T(t-s) [g(s, Cx(s)) - g(s, Cy(s))] ds \right\|_G \\ &\quad + \left\| \int_{-\infty}^t T(t-s) [\sigma(s, Dx(s)) - \sigma(s, Dy(s))] dW(s) \right\|_G. \end{aligned}$$

Since $(a + b + c + d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$, we have

$$\begin{aligned} E\|(\Gamma x)(t) - (\Gamma y)(t)\|_G^2 &\leq 4E\|f(t, Bx(t)) - f(t, By(t))\|_G^2 \\ &\quad + 4E\left\| \int_{-\infty}^t AT(t-s) [f(s, Bx(s)) - f(s, By(s))] ds \right\|_G^2 \\ &\quad + 4E\left\| \int_{-\infty}^t T(t-s) [g(s, Cx(s)) - g(s, Cy(s))] ds \right\|_G^2 \\ &\quad + 4E\left\| \int_{-\infty}^t T(t-s) [\sigma(s, Dx(s)) - \sigma(s, Dy(s))] dW(s) \right\|_G^2. \end{aligned}$$

We first evaluate the first term of the right-hand side as follows:

$$E\|f(t, Bx(t)) - f(t, By(t))\|^2 \leq h^2 L_f \|x - y\|_{G, \infty}^2.$$

As to the second term, it follows from (H1) that

$$\begin{aligned} &E\left\| \int_{-\infty}^t AT(t-s) [f(s, Bx(s)) - f(s, By(s))] ds \right\|_G^2 \\ &\leq E\left[\int_{-\infty}^t e^{-\omega(t-s)} \gamma(t-s) \|f(s, Bx(s)) - f(s, By(s))\| ds \right]^2 \\ &\leq \left(\int_{-\infty}^t e^{-\omega(t-s)} \gamma(t-s) ds \right)^2 \sup_{t \in R} E\|f(s, Bx(s)) - f(s, By(s))\|^2 \\ &\leq \rho^2 h^2 L_f \|x - y\|_{G, \infty}^2. \end{aligned}$$

As to the third term, we obtain that

$$\begin{aligned} &E\left\| \int_{-\infty}^t T(t-s) [g(s, Cx(s)) - g(s, Cy(s))] ds \right\|_G^2 \\ &\leq M^2 E\left\| \int_{-\infty}^t e^{-\delta(t-s)} [g(s, Cx(s)) - g(s, Cy(s))] ds \right\|_G^2 \\ &\leq h^2 \frac{1}{\delta^2} M^2 L_g \|x - y\|_{G, \infty}^2. \end{aligned}$$

As far as the last term is concerned, we use an estimate on the Itô integral established in [21, Proposition 1.9] and obtain:

$$\begin{aligned} &E\left\| \int_{-\infty}^t T(t-s) [\sigma(s, Dx(s)) - \sigma(s, Dy(s))] dW(s) \right\|_G^2 \\ &\leq M^2 E\left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right) \int_{-\infty}^t e^{-\delta(t-s)} \|\sigma(s, Dx(s)) - \sigma(s, Dy(s))\|^2 ds \\ &\leq h^2 \frac{1}{\delta^2} M^2 L_\sigma \|x - y\|_{G, \infty}^2. \end{aligned}$$

Thus, by combining, it follows that

$$\|\Gamma x - \Gamma y\|_{G,\infty} \leq 4 \left[h^2 L_f + \rho^2 h^2 L_f + h^2 \frac{1}{\delta^2} M^2 L_g + h^2 \frac{1}{\delta^2} M^2 L_\sigma \right] \|x - y\|_{G,\infty},$$

which shows that $\Gamma(\cdot)$ is a contraction and completes the proof of this result. $\square$

When $f = 0$, $B = C = D = I$, system (1) reduces to the following stochastic system

$$dx(t) = (Ax(t) + g(t, x(t))) dt + \sigma(t, x(t)) dW(t), \quad t \in R, \quad (5)$$

Corollary 1. Assume the conditions (H1) – (H3) hold. Then (5) has a unique square-mean almost period mild solution on R provided that

$$2 \left[\frac{1}{\delta^2} M^2 L_g + \frac{1}{\delta^2} M^2 L_\sigma \right] < 1. \quad (6)$$

Proof. The proof is similar the proof of Theorem 3.1, so we omit it here. $\square$

Remark 3.1. Corollary 1 in the case $G = H$ is the main result of [9].

Theorem 3.2. Assume the conditions (H1) and (H4) – (H6) hold. Then (1) has a unique square-mean almost automorphic mild solution on R provided that

$$4 \left[h^2 L_f + \rho^2 h^2 L_f + h^2 \frac{1}{\delta^2} M^2 L_g + h^2 \frac{1}{\delta^2} M^2 L_\sigma \right] < 1. \quad (7)$$

Proof. Let $\Gamma: AA(R; L^2(P; G)) \rightarrow AA(R; L^2(P; G))$ be the map defined by

$$\begin{aligned} \Gamma x(t) = & -f(t, x(t-r)) - \int_{-\infty}^t AT(t-s)f(s, x(s-r))ds \\ & + \int_{-\infty}^t T(t-s)g(s, x(s-r))ds + \int_{-\infty}^t T(t-s)\sigma(s, x(s-r))dW(s), \quad t \in \mathbb{R} \end{aligned} \quad (8)$$

From Lemma 2.5, Lemma 3.4, Lemma 3.5 and Lemma 3.6, it follows that the nonlinear operator Γ is well defined, that is, it maps $AA(R; L^2(P; G)) \rightarrow AA(R; L^2(P; G))$. As the discussion in Theorem 3.1, we can prove that Γ is a contraction. So, the proof is completed. $\square$

Corollary 2. Assume the conditions (H1), (H5) and (H6) hold. Then (5) has a unique square-mean automorphic mild solution on R provided that

$$2 \left[\frac{1}{\delta^2} M^2 L_g + \frac{1}{\delta^2} M^2 L_\sigma \right] < 1. \quad (9)$$

Proof. The proof is similar the proof of Theorem (3.1), so we omit it here. $\square$

Remark 3.2. Corollary 2 in the case $G = H$ is the main result of [17].

4 Example

In this section an example is presented to illustrate Theorem 3.1. We consider the following neutral stochastic equation.

$$\frac{\partial}{\partial t} x(t, \xi) = \frac{\partial^2}{\partial \xi^2} x(t, \xi) - \frac{\partial}{\partial t} f(t, Bx(t, \xi)) + g(t, Cx(t, \xi)) + \sigma(t, Dx(t, \xi)) dW(t) \quad (10)$$

$$0 \leq \xi \leq 1.$$

with the Dirichlet boundary conditions

$$x(t, 0) = x(t, 1) = 0. \quad (11)$$

Following a well-known approach of [22] we will rewrite system (10) and (11) as an evolution equation

$$d[x(t) + f(t, Bx(t))] = (Ax(t) + g(t, Cx(t))) dt + \sigma(t, Dx(t)) dW(t) \quad (12)$$

in the space $H = L^2[0, 1]$, $W(t)$ stands for a standard cylindrical Wiener process in H defined in a stochastic basis $(\Omega, \mathcal{F}, P)$. Let $A : H \rightarrow H$ be the linear operator defined by $A = (\partial^2 / \partial \xi^2)$, where $D(A) = \{y \in H : y, dy/dx \text{ are absolutely continuous, and } (d^2/dx^2)y \in H, y(0) = y(1) = 0\}$.

Then A generates a symmetric C_0 -semigroup e^{-tA} in H and so that there exists an $M > 0$ and $\delta > 0$ such that $\|e^{-tA}\| \leq Me^{-\delta t}$. Also note that there exists a complete orthonormal set $\{\zeta_n\}$ ($n = 1, 2, 3, \dots$) of eigenvectors of A with $\zeta_n(x) = \sqrt{2/\pi} \sin nx$. Then, the operator $A^{-\frac{1}{2}}$ is given by $A^{-\frac{1}{2}}\zeta = \sum_{n=1}^{\infty} n \langle \zeta, \zeta_n \rangle \zeta_n$ on the space $D[A^{\frac{1}{2}}] = \left\{ \zeta(\cdot) \in H : \sum_{n=1}^{\infty} n \langle \zeta, \zeta_n \rangle \zeta_n \in H \right\}$ (for more details see [22]).

For the sake of simplicity, we consider the case where $G = H = L^2[0, 1]$ and $B = C = D$ are the bounded multiplication operator defined by: $Bx = Cx = Dx = q(x)$ for each $u \in L^2[0, 1]$ where $q \in L^\infty[0, 1]$. In this even, $\|B\|_{L(L^2(P; H))} = \|C\|_{L(L^2(P; H))} = \|D\|_{L(L^2(P; H))} = \|q\|_\infty$.

Thus, under assumptions (H1) – (H4), once (3) holds, an application of Theorem 3.1 yields that (1) has a unique mild solution, which is obviously square-mean almost periodic.

Thus, under assumptions (H1) and (H4) – (H6), once (7) holds, an application of Theorem 3.2 yields that (1) has a unique mild solution, which is obviously square-mean almost automorphic.

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