Ruled Surface Pair Generated by a Curve and its Natural Lift in $\mathbb{R}^3$

Evren ERGÜN

Ondokuz Mayıs University Faculty of Arts and Sciences
Department of Mathematics, Samsun, Turkey
eergun@omu.edu.tr

Mustafa ÇALIŞKAN

Gazi University Faculty of Sciences
Department of Mathematics, Ankara, Turkey
mustafacaliskan@gazi.edu.tr

Abstract

In this study, firstly, the Frenet vector fields $T, N, B$ of the natural lift $\overline{\alpha}$ of a curve $\alpha$ are calculated in terms of those of $\alpha$ in $\mathbb{R}^3$. Secondly, we obtained striction lines and distribution parameters of ruled surface pair generated by the curve $\alpha$ and its natural lift $\overline{\alpha}$. Finally, for $\alpha$ and $\overline{\alpha}$ those notions are compared with each other.

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1 Introduction and Preliminary Notes

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha$, where $T$, $N$ and $B$ are the tangent, the principal normal and the binormal vector fields of the curve $\alpha$, respectively.

Let $\alpha$ be a regular curve in $\mathbb{R}^3$. Then

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N = B \times T, \quad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}. \quad [4].$$

If $\alpha$ is a unit speed curve, then

$$T = \alpha', \quad N = \frac{\alpha''}{\|\alpha''\|}, \quad B = T \times N, \quad [4].$$
Let \( \alpha \) be a unit speed space curve with curvature \( \kappa \) and torsion \( \tau \). Let Frenet vector fields of \( \alpha \) be \( \{T, N, B\} \). Then, Frenet formulas are given by
\[
T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, [1].
\]
where \( \kappa = \langle T', N \rangle \) and \( \tau = \langle N', B \rangle \).

For any unit speed curve \( \alpha : I \rightarrow \mathbb{R}^3 \), we call \( W(s) = \tau T(s) + \kappa B(s) \) the Darboux vector field of \( \alpha \).[1]

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation
\[
X(s, v) = \alpha(s) + ve(s), [4].
\]
where \( \alpha(s) \) represents a space curve which is called the base curve and \( e \) is a unit vector representing the direction of a straight line.

The striction point on a ruled surface \( X \) is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given as
\[
\beta(s) = \alpha(s) - \frac{\langle \alpha', e' \rangle}{\langle e', e' \rangle} e(s). [4].
\]
The distribution parameter of the ruled surface \( X \) is defined by
\[
P_e = \frac{\det \left( \alpha', e, e' \right)}{\|e'\|^2} [4].
\]
The ruled surface is developable if and only if \( P_e = 0 \).

Let \( M \) be a hypersurface in \( \mathbb{R}^3 \) and let \( \alpha : I \rightarrow M \) be a parametrized curve. \( \alpha \) is called an integral curve of \( X \) if
\[
\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \quad (\text{for all } t \in I), [1].
\]
where \( X \) is a smooth tangent vector field on \( M \). We have
\[
TM = \bigcup_{P \in M} T_PM = \chi(M)
\]
where \( T_PM \) is the tangent space of \( M \) at \( P \) and \( \chi(M) \) is the space of vector fields on \( M \).

For any parametrized curve \( \alpha : I \rightarrow M, \bar{\alpha} : I \rightarrow TM \) given by
\[
\bar{\alpha}(s) = \left( \alpha(s), \alpha'(s) \right) = \alpha'(s)|_{\alpha(s)}, [2]
\]
is called the natural lift of \( \alpha \) on \( TM \). Thus, we can write
\[
\frac{d\bar{\alpha}}{ds} = \frac{d}{ds} (\alpha'(s)|_{\alpha(s)}) = D_{\alpha'(s)} \alpha'(s)
\]
where \( D \) is the Levi-Civita connection on \( \mathbb{R}^3 \).
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For any parametrized curve $\alpha : I \rightarrow M$ in $\mathbb{R}^3$, $\overline{\alpha} : I \rightarrow TM$ given by

$$\overline{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of $\alpha$ on $TM$.

We denote by $\{\overline{T}(s), \overline{N}(s), \overline{B}(s)\}$ the moving Frenet frame along the curve $\overline{\alpha}$, where $\overline{T}, \overline{N}$ and $\overline{B}$ are the tangent, the principal normal and the binormal vector fields of the curve $\overline{\alpha}$, respectively.

Now we have

$$\overline{T}(s) = \frac{\overline{\alpha}'}{||\overline{\alpha}'||} = \frac{\alpha'''}{||\alpha'''||} = N(s)$$

$$\overline{B}(s) = \frac{\overline{\alpha} \times \overline{\alpha}'''}{||\overline{\alpha} \times \overline{\alpha}'''||} = \frac{\alpha'' \times \alpha'''}{||\alpha'' \times \alpha'''||} = \frac{\kappa N \times (\kappa' N + \kappa N'')}{||\kappa N \times (\kappa' N + \kappa N'')||}$$

$$= \frac{k^2 \tau}{k^2 \sqrt{k^2 + \tau^2}} T(s) + \frac{k^3}{k^2 \sqrt{k^2 + \tau^2}} B(s) = \frac{\tau}{||W||} T(s) + \frac{\kappa}{||W||} B(s)$$

$$\overline{N}(s) = \left(\frac{\tau}{||W||} T(s) + \frac{\kappa}{||W||} B(s)\right) \times N(s) = -\frac{\kappa}{||W||} T(s) + \frac{\tau}{||W||} B(s).$$

**Corollary 1** Let $\overline{\alpha}$ be a regular curve in $\mathbb{R}^3$. Then

$$\overline{T}(s) = N(s)$$

$$\overline{N}(s) = -\frac{\kappa(s)}{||W||} T(s) + \frac{\tau(s)}{||W||} B(s)$$

$$\overline{B}(s) = \frac{\tau(s)}{||W||} T(s) + \frac{\kappa(s)}{||W||} B(s).$$

(i) Let $X$ and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vT(s), \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{T}(s)$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{T}(s)$, respectively. Now let us calculate $\lambda$ and $\mu$.
\[ \lambda = \left\langle \alpha', T' \right\rangle = \left\langle T, \kappa N \right\rangle = 0, \quad \mu = \left\langle \pi', T' \right\rangle = \left\langle \kappa N, -\kappa T + \tau B \right\rangle = 0 \]

The distribution parameters of the ruled surfaces \( X \) and \( \overline{X} \) are defined by
\[ P_T = \frac{\text{det} \left( \alpha', T, T' \right)}{\| T' \|^2} \quad \text{and} \quad \overline{P}_T = \frac{\text{det} \left( \pi', T, T' \right)}{\| T' \|^2}. \]

Then we have
\[ P_T = \frac{\text{det} \left( \alpha', T, T' \right)}{\| T' \|^2} = \frac{\text{det} \left( T, T, \kappa N \right)}{\| T \|^2} = 0 \]
\[ \overline{P}_T = \frac{\text{det} \left( \alpha', T, T' \right)}{\| T' \|^2} = \frac{\text{det} \left( \kappa N, N, -\kappa T + \tau B \right)}{\| T' \|^2} = 0. \]

**Corollary 2** Let the striction curves of \( X \) and \( \overline{X} \) be given by \( \beta(s) = \alpha(s) - \lambda T(s) \) and \( \overline{\beta}(s) = \overline{\alpha}(s) - \mu T(s) \), respectively. Then \( \beta(s) = \alpha(s) \) and \( \overline{\beta}(s) = \overline{\alpha}(s) \).

**Corollary 3** If the ruled surface \( X \) is developable then the ruled surface \( \overline{X} \) are also developable.

(ii) Let \( X \) and \( \overline{X} \) be two ruled surfaces which is given by
\[ X(s, v) = \alpha(s) + vN(s), \quad \overline{X}(s, v) = \pi(s) + v\overline{N}(s) \]

The striction curves of \( X \) and \( \overline{X} \) are given by \( \beta(s) = \alpha(s) - \lambda N(s) \) and \( \overline{\beta}(s) = \pi(s) - \mu \overline{N}(s) \), respectively. Now let us calculate \( \lambda \) and \( \mu \)
\[ \lambda = \left\langle \alpha', N' \right\rangle = \frac{\left\langle T, -\kappa T + \tau B \right\rangle}{\kappa^2 + \tau^2} = \frac{-\kappa}{\kappa^2 + \tau^2} \]
\[ \mu = \left\langle \pi', \overline{N} \right\rangle = \left\langle \kappa N, -\frac{s'}{\| W \|} T - \frac{s'}{\| W \|} T + \frac{s'''}{\| W \|} B + \frac{s''}{\| W \|} B \right\rangle = \left\langle \kappa N, -\frac{s'}{\| W \|} T - \frac{s'}{\| W \|} + \frac{s''}{\| W \|} B + \frac{s'''}{\| W \|} (-\tau N) \right\rangle \]
\[ = \left\langle \kappa N, -\frac{s'}{\| W \|} T - \frac{s''}{\| W \|} N + \frac{s'''}{\| W \|} B \right\rangle = \frac{-\kappa (s''^2 + s''')^2}{\| W \|^2} = \frac{-\kappa || W ||^3}{\kappa^2 + \tau^2 + || W ||^2}. \]

The distribution parameters of the ruled surfaces \( X \) and \( \overline{X} \) are defined by
\[ P_N = \frac{\text{det} \left( \alpha', N', N' \right)}{\| N' \|^2} \quad \text{and} \quad \overline{P}_N = \frac{\text{det} \left( \pi', \overline{N}, \overline{N} \right)}{\| \overline{T} \|^2}. \]

Then we obtain
\[ P_N = \frac{\text{det} \left( \alpha', N, N' \right)}{\| N' \|^2} = \frac{\text{det} \left( T, N, -\kappa T + \tau B \right)}{\| (-\kappa T + \tau B) \|^2} \]
\[ = \frac{\text{det} \left( T, N, -\kappa T \right) + \text{det} \left( T, N, \tau B \right)}{\kappa^2 + \tau^2} = \frac{\text{det} \left( T, N, \tau B \right)}{\kappa^2 + \tau^2} \]
\[ = \frac{\left\langle T, N \times \tau B \right\rangle}{\kappa^2 + \tau^2} = \tau \]
Corollary 5 Let the striction curves of \(X\) and \(\overline{X}\) be given by \(\beta(s) = \alpha(s) - \lambda N(s)\) and \(\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)\), respectively. Then \(\mu = \frac{-\kappa\left(-\frac{1}{\kappa}\right)^2}{\kappa^2 + \tau^2 + (\kappa^2 + \tau^2)^2}\).

Corollary 5 Let the distribution parameters of the ruled surfaces \(X\) and \(\overline{X}\) be \(P_N\) and \(\overline{P}_N\), respectively. Then \(\overline{P}_N = \frac{\kappa^2 + \kappa\kappa'}{\kappa^2 + \tau^2 + (\kappa_\beta)^2}\).

(iii) Let \(X\) and \(\overline{X}\) be two ruled surfaces which are given by

\[X(s,v) = \alpha(s) + vB(s), \overline{X}(s,v) = \overline{\alpha}(s) + \overline{vB}(s)\]

The striction curves of \(X\) and \(\overline{X}\) are given by \(\beta(s) = \alpha(s) - \lambda B(s)\) and \(\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)\), respectively. Now let us calculate \(\lambda\) and \(\mu\)

\[\lambda = \frac{\langle \alpha', B' \rangle}{\langle B', B' \rangle} = \frac{\langle T, \tau N \rangle}{\langle B', B' \rangle} = 0\]

\[\mu = \frac{\langle \overline{\alpha}', \overline{B}' \rangle}{\langle \overline{B}', \overline{B}' \rangle} = \frac{\langle \kappa N, -\kappa T + \frac{\kappa'}{\|W\|} B, -\kappa T + \frac{\kappa'}{\|W\|} B' + \frac{\kappa'}{\|W\|} B' \rangle}{\langle \overline{B}', \overline{B}' \rangle} = 0\]

The distribution parameters of the ruled surfaces \(X\) and \(\overline{X}\) are defined by \(P_B = \frac{\det(\alpha', B, B')}{\|B'\|^2}\) and \(\overline{P}_B = \frac{\det(\overline{\alpha}', \overline{B}, \overline{B}')}{\|\overline{B}'\|^2}\). Now we get

\[P_B = \frac{\det(\alpha', B, B')}{\|B'\|^2} = \frac{\det(T, B, -\tau N)}{\|B'\|^2} = \frac{\langle T, B \times (-\tau N) \rangle}{\tau^2} = \frac{\tau}{\tau^2} = \frac{1}{\tau}\]
\[
\overline{P_B} = \frac{\det \left( \alpha', B, B' \right)}{\|B\|^2} = \frac{\det \left( \kappa N, \frac{\tau}{\|W\|}T + \frac{\kappa}{\|W\|}B, \frac{\tau'}{\|W\|}T + \frac{\kappa'}{\|W\|}B + \frac{\kappa}{\|W\|} \right)}{\|B\|^2}
\]
\[
= \frac{\det \left( \kappa N, \frac{\tau}{\|W\|}T + \frac{\kappa}{\|W\|}B, \frac{\tau'}{\|W\|}T + \frac{\kappa N}{\|W\|} + \frac{\kappa'}{\|W\|}B + \frac{\kappa}{\|W\|} - \tau N \right)}{\|B\|^2}
\]
\[
= \frac{\frac{\kappa^2 \tau' - \kappa \kappa'}{\|W\|^2}}{\kappa'^2 + \tau'^2} = \frac{\kappa^2 \tau' - \kappa \kappa'}{\kappa'^2 + \tau'^2}
\]

**Corollary 6** Let the striction curves of \( X \) and \( \overline{X} \) be given by \( \beta(s) = \alpha(s) - \lambda B(s) \) and \( \overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s) \), respectively. Then \( \beta(s) = \alpha(s) \) and \( \overline{\beta}(s) = \overline{\alpha}(s) \).

**Corollary 7** Let the distribution parameters of the ruled surfaces \( X \) and \( \overline{X} \) be \( P_B \) and \( \overline{P_B} \), respectively. Then \( \overline{P_B} = \frac{\kappa^2 \tau' - \kappa \kappa'}{\kappa'^2 + \tau'^2} \).

**References**


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