On Natural Lift of a Curve

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Abstract
In this study, the Frenet vector fields $\overline{T}, \overline{N}, \overline{B}$, curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ of the natural lift $\overline{\alpha}$ of a curve $\alpha$ are calculated in terms of those of $\alpha$ in $\mathbb{R}^3$. The same study has been done in $\mathbb{R}^4$.

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1 Introduction and Preliminary Notes

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha$, where $T$, $N$ and $B$ are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively.

Let $\alpha$ be a regular curve in $\mathbb{R}^3$. Then

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N = B \times T, \quad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad [6].$$

If $\alpha$ is a unit speed curve, then

$$T = \alpha', \quad N = \frac{\alpha''}{\|\alpha''\|}, \quad B = T \times N, \quad [6].$$
Let $\alpha$ be a unit speed space curve with curvature $\kappa$ and torsion $\tau$ and let Frenet vector fields of $\alpha$ be $\{T, N, B\}$. Then, Frenet formulas are given by

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N,$$

where $\kappa = \langle T', N \rangle$ and $\tau = \langle N', B \rangle$.

For any unit speed curve $\alpha : I \rightarrow \mathbb{R}^3$, we call $W(s) = \tau T(s) + \kappa B(s)$ the Darboux vector field of $\alpha$, [1].

Let $M$ be a hypersurface in $\mathbb{R}^3$ and let $\alpha : I \rightarrow M$ be a parametrized curve. $\alpha$ is called an integral curve of $X$ if

$$d\frac{ds}{ds}(\alpha(s)) = X(\alpha(s)) \quad \text{(for all } s \in I), [1].$$

where $X$ is a smooth tangent vector field on $M$. We have $TM = \bigcup_{P \in M} T_PM = \chi(M)$, where $T_PM$ is the tangent space of $M$ at $P$ and $\chi(M)$ is the space of vector fields on $M$.

For any parametrized curve $\alpha : I \rightarrow M$, $\overline{\alpha} : I \rightarrow TM$ given by

$$\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}, [5].$$

is called the natural lift of $\alpha$ on $TM$. Thus, we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds} \left( \alpha'(s)|_{\alpha(s)} \right) = D_{\alpha'(s)}\alpha'(s)$$

where $D$ is the Levi-Civita connection on $\mathbb{R}^3$.

Let $\alpha : I \rightarrow \mathbb{R}^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha$, where $T, N$ and $B$ are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively.

Let $\alpha$ be a unit speed timelike space curve with curvature $\kappa$ and torsion $\tau$. Let Frenet vector fields of $\alpha$ be $\{T, N, B\}$. In this trihedron, $T$ is timelike vector field, $N$ and $B$ are spacelike vector fields. Then, Frenet formulas are given by [4]

$$T' = \kappa N \quad N' = \kappa T + \tau B \quad B' = -\tau N.$$

Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that $T$ and $B$ are spacelike vector fields and $N$ is a timelike vector field. Then, Frenet formulas are given by [4]

$$T' = \kappa N \quad N' = \kappa T + \tau B \quad B' = \tau N.$$

Let $\alpha$ be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that $T$ and $N$ are spacelike vector fields and $B$ is a timelike vector field. Then, Frenet formulas are given by [4]
\[ T' = \kappa N \ N' = -\kappa T + \tau B \ B' = \tau N. \]

Let \( M \) be a hypersurface in \( \mathbb{R}^3 \) and let \( \alpha : I \longrightarrow M \) be a parametrized curve. \( \alpha \) is called an integral curve of \( X \) if
\[
\frac{d}{ds} (\alpha (s)) = X (\alpha (s)) \text{ (for all } s \in I) \]
where \( X \) is a smooth tangent vector field on \( M \). We have
\[
TM = \bigcup_{P \in M} T_P M = \chi (M)
\]
where \( T_P M \) is the tangent space of \( M \) at \( P \) and \( \chi (M) \) is the space of vector fields on \( M \).

For any parametrized curve \( \alpha : I \longrightarrow M \), \( \overline{\alpha} : I \longrightarrow TM \) given by
\[
\overline{\alpha} (s) = \left( \alpha (s), \alpha' (s) \right) = \alpha' (s) |_{\alpha(s)}. \]

is called the natural lift of \( \alpha \) on \( TM \). Thus, we can write
\[
\frac{d}{ds} (\overline{\alpha}) = \frac{d}{ds} (\alpha' |_{\alpha(s)}) = D_{\alpha'(s)} \alpha' (s)
\]
where \( D \) is the Levi-Civita connection on \( \mathbb{R}^3 \).

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For any parametrized curve in \( \mathbb{R}^3 \) \( \alpha : I \longrightarrow M \), \( \overline{\alpha} : I \longrightarrow TM \) given by
\[
\overline{\alpha} (s) = \left( \alpha (s), \alpha' (s) \right) = \alpha' (s) |_{\alpha(s)}
\]

is called the natural lift of \( \alpha \) on \( TM \).

We denote by \( \{ \overline{T} (s), \overline{N} (s), \overline{B} (s) \} \) the moving Frenet frame along the curve \( \overline{\alpha} \), where \( \overline{T}, \overline{N} \) and \( \overline{B} \) are the tangent, the principal normal and the binormal vector of the curve \( \overline{\alpha} \), respectively.

**Corollary 1** Let \( \overline{\alpha} \) be the natural lift of \( \alpha \) in \( \mathbb{R}^3 \) and be a regular curve. Then
\[
\begin{align*}
\overline{T} (s) &= N (s) \\
\overline{N} (s) &= -\frac{\kappa (s)}{\|W\|} T (s) + \frac{\tau (s)}{\|W\|} B (s) \\
\overline{B} (s) &= \frac{\tau (s)}{\|W\|} T (s) + \frac{\kappa (s)}{\|W\|} B (s).
\end{align*}
\]
Let $\alpha$ be a space curve with curvature $\kappa$ and torsion $\tau$. Then $\pi = \langle T', N \rangle$ and $\tau = \langle N', B \rangle$.

**Corollary 2** Let $\alpha$ be the natural lift of $\alpha$ with curvature $\kappa$ and torsion $\tau$. Then

$$\kappa(s) = \frac{\kappa^2(s) + \tau^2(s)}{\|W\|}, \quad \tau(s) = \frac{-\kappa'(s) \tau(s) + \kappa(s) \tau'(s)}{\|W\|^2}.$$ 

For any parametrized curve in $\mathbb{R}^3$ $\alpha : I \rightarrow M$, $\pi : I \rightarrow TM$ given by $\pi(s) = \left( \alpha(s), \alpha'(s) \right) = \alpha'(s)|_{\alpha(s)}$ is called the natural lift of $\alpha$ on $TM$.

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha$, where $T, N$ and $B$ are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively.

**Corollary 3** Let $\alpha$ be a unit speed timelike space curve and $\pi$ be the natural lift of $\alpha$. Then

$$\begin{align*}
T(s) &= N(s) \\
N(s) &= -\frac{\kappa(s)}{\|W\|} T(s) - \frac{\tau(s)}{\|W\|} B(s) \\
B(s) &= -\frac{\tau(s)}{\|W\|} T(s) - \frac{\kappa(s)}{\|W\|} B(s).
\end{align*}$$

**Corollary 4** Let $\alpha$ be a unit speed timelike space curve and the natural lift $\pi$ of the curve $\alpha$ be a space curve with curvature $\kappa$ and torsion $\tau$. Then

$$\begin{align*}
\kappa(s) &= \frac{\kappa^2(s) - \tau^2(s)}{\|W\|}, \quad \tau(s) = \frac{-\kappa'(s) \tau(s) + \kappa(s) \tau'(s)}{\|W\|^2}.
\end{align*}$$

**Corollary 5** Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal and $\pi$ be the natural lift of $\alpha$. Then

$$\begin{align*}
T(s) &= N(s) \\
N(s) &= \frac{\kappa(s)}{\|W\|} T(s) + \frac{\tau(s)}{\|W\|} B(s) \\
B(s) &= \frac{\tau(s)}{\|W\|} T(s) - \frac{\kappa(s)}{\|W\|} B(s).
\end{align*}$$
Corollary 6 Let \( \alpha \) be a unit speed spacelike space curve with a spacelike binormal and the natural lift \( \overline{\alpha} \) of the curve \( \alpha \) be a space curve with curvature \( \overline{\kappa} \) and torsion \( \overline{\tau} \). Then

\[
\overline{\kappa}(s) = \frac{\kappa^2(s) + \tau^2(s)}{\|W\|}, \quad \overline{\tau}(s) = \frac{\kappa'(s)\tau(s) - \kappa(s)\tau'(s)}{\|W\|^2}.
\]

Corollary 7 Let \( \alpha \) be a unit speed spacelike space curve with a timelike binormal and \( \overline{\alpha} \) be the natural lift of \( \alpha \). Then

\[
\overline{T}(s) = N(s), \\
\overline{N}(s) = -\frac{\kappa(s)}{\|W\|}T(s) - \frac{\tau(s)}{\|W\|}B(s), \\
\overline{B}(s) = \frac{\tau(s)}{\|W\|}T(s) + \frac{\kappa(s)}{\|W\|}B(s).
\]

Corollary 8 Let \( \alpha \) be a unit speed spacelike space curve with a timelike binormal and the natural lift \( \overline{\alpha} \) of the curve \( \alpha \) be a space curve with curvature \( \overline{\kappa} \) and torsion \( \overline{\tau} \). Then

\[
\overline{\kappa}(s) = \frac{\kappa^2(s) + \tau^2(s)}{\|W\|}, \quad \overline{\tau}(s) = \frac{-\kappa'(s)\tau(s) + \kappa(s)\tau'(s)}{\|W\|^2}.
\]

References


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