Homothetic Exponential Motions with Generalized Quaternions

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Abstract

In this paper, we give the real matrix representations of generalized quaternions, and then, the concept of homothetic exponential motions is discussed and their velocities, accelerations obtained, Also, the paper gives some formula and facts about exponential motions which are not generally known for generalized quaternions.

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1 Introduction

To state the geometry of the motion of a point in the motion of space is significant in the study of kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. Hacısalıhoğlu [1], [2] showed some properties of 1-parameter homothetic motion in Euclidean space E^n . Subsequently, Kula and Yaylı [3], and [4] expressed Hamilton motions by means of Hamilton operators in semi-Euclidean spaces E_2^4 and showed that this motions, are a homothetic motion. In this paper, first, we give 4x4 matrices representations of the generalized quaternions and of their algebraic properties. Furthermore, we

define that a motions by using these matrices, and show that this motion is a homothetic exponential motions. We show that the homothetic exponential motions has only one pole point at every instant t , and define that this motion has only one acceleration center of high order at every instant t . We hope that these results will contribute to the study of space kinematics and physics.

2 Generalized Quaternions

A generalized quaternion is a number of form

$$\omega = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3 \quad (1)$$

where $\omega_0, \omega_1, \omega_2, \omega_3$ are reel numbers called the components of ω , and i, j, k are arbitrary "units" which satisfy the relations

$$\left\{ \begin{array}{l} i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta, \quad ij = -ji = k \\ jk = \beta i = -kj, \quad ki = \alpha j = -ik; \quad \alpha, \beta \in R. \end{array} \right\} \quad (2)$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$ and

$$H_{\alpha\beta} = \{\omega \mid \omega = \omega_0 + \omega_1 i + \omega_2 j + \omega_3 k; \omega_0, \omega_1, \omega_2, \omega_3 \in R\}. \quad (3)$$

Write $\omega_0 + \omega_1 i + \omega_2 j + \omega_3 k$ as $(\omega_0 + \omega_1 i) + j(\omega_2 + \omega_3 i) = z_1 + jz_2$ where z_1, z_2 are complex numbers" C

Furthermore, rules of multiplication and addition of two complex numbers have the following properties [7].

$$\left\{ \begin{array}{l} (z_1 + jz_2) \cdot (z_3 + jz_4) = (z_1 z_3 - z_2 z_4) + j(z_1 z_4 + z_2 z_3), \\ (z_1 + jz_2) + (z_3 + jz_4) = (z_1 + z_3) + j(z_2 + z_4), \\ z_1 + jz_2 = z_3 + jz_4 \iff z_1 = z_3 \text{ and } z_2 = z_4. \end{array} \right\} \quad (4)$$

In particular, in eq.[4], if we put $z_1 = a$ and $z_2 = ib$ with $a, b \in C$. In that case, $H_{\alpha\beta}$ is a algebra and non -associative. This algebra is called generalized quaternion algebra and are denoted by $H_{\alpha\beta}$, Note that one of the basis of this algebra is $\{1, i, j, k\}$ and the dimension is 4. It is clear, if $\alpha = \beta = 1$ then $H_{\alpha\beta}$ real quaternion, and if $\alpha = 1, \beta = -1$ then $H_{\alpha\beta}$ split quaternion.

Definition 2.1 (*The addition and multiplication*): Addition of two generalized quaternions $u, w \in H_{\alpha\beta}$ is defined by

$$u + w = (u_0 + w_0) + (u_1 + w_1)i + (u_2 + w_2)j + (u_3 + w_3)k, \quad (5)$$

whereas mutiplication is defined by

$$\left\{ \begin{array}{l} u \cdot w = (u_0 w_0 - \alpha u_1 w_1 - \beta u_2 w_2 - \alpha \beta u_3 w_3) + \\ (u_0 w_1 + u_1 w_0 + \beta u_2 w_3 - \beta u_3 w_2)i + \\ (u_0 w_2 + u_2 w_0 + \alpha u_3 w_1 - \alpha u_1 w_3)j + \\ (u_0 w_3 + u_1 w_2 - u_2 w_1 + u_3 w_0)k. \end{array} \right\} \quad (6)$$

Definition 2.2 (*The Concept of Conjugate for Generalized Quaternions*): The conjugate of ω is defined to be

$$\overline{\omega} = \omega_0 - \omega_1 i - \omega_2 j - \omega_3 k. \quad (7)$$

Definition 2.3 (*Norms of Generalized Quaternions*): The norm of ω is defined to be

$$|\omega| = \sqrt{\omega \overline{\omega}} = \sqrt{\omega_0^2 + \alpha \omega_1^2 + \beta \omega_2^2 + \alpha \beta \omega_3^2}, \quad (8)$$

$$N_\omega = \omega_0^2 + \alpha \omega_1^2 + \beta \omega_2^2 + \alpha \beta \omega_3^2. \quad (9)$$

Although $H_{\alpha\beta}$ is nonassociative, it is still an alternative, flexible and division algebra over $\mathbf{R}$, that is, for all $u, w \in H_{\alpha\beta}$, the following equalities hold:

1. $u(u.\omega) = u^2.\omega$
2. $(u\omega).\omega = u.\omega^2$
3. $u(u.\omega) = (u\omega).\omega$
4. $|u.\omega| = |u| |\omega|$
5. $\omega^{-1} = \frac{\overline{\omega}}{|\omega|^2}$

For the generalized quaternion algebra $H_{\alpha\beta}$. It is well known that through the bijective map [5], [6]

$$\phi : \omega = \omega_0 + \omega_1 i + \omega_2 j + \omega_3 k \in H_{\alpha\beta} \longrightarrow \phi(\omega) = \begin{bmatrix} \omega_0 & -\alpha\omega_1 & -\beta\omega_2 & -\alpha\beta\omega_3 \\ \omega_1 & \omega_0 & -\beta\omega_3 & \beta\omega_2 \\ \omega_2 & \alpha\omega_3 & \omega_0 & -\alpha\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & \omega_0 \end{bmatrix}, \quad (10)$$

$H_{\alpha\beta}$ is algebraically isomorphic to the matrix algebra

$$M = \left\{ \begin{bmatrix} \omega_0 & -\alpha\omega_1 & -\beta\omega_2 & -\alpha\beta\omega_3 \\ \omega_1 & \omega_0 & -\beta\omega_3 & \beta\omega_2 \\ \omega_2 & \alpha\omega_3 & \omega_0 & -\alpha\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & \omega_0 \end{bmatrix} : \omega_{0-3}, \alpha, \beta \in R \right\} \quad (11)$$

and $\phi(\omega)$ is a faithful real matrix representation of ω .

Lemma 2.4 Let $u, \omega \in H_{\alpha\beta}$, $k \in R$, [6].

1. $u = \omega \Leftrightarrow \phi(u) = \phi(\omega)$

2. $\phi(u + \omega) = \phi(u) + \phi(\omega)$
3. $\phi(u.\omega) = \phi(u).\phi(\omega)$
4. $\phi(k.\omega) = k.\phi(\omega); \quad \phi(1) = I_4$
5. $\phi(\overline{\omega}) = \phi^T(\omega)$

Lemma 2.5 *Let $\omega = \omega_0 + \omega_1 i + \omega_2 j + \omega_3 k \in H_{\alpha\beta}$, and denote $\omega = [\omega_0, \omega_1, \omega_2, \omega_3]^T$, called the vector representation of ω . Then for all $u, \omega \in H_{\alpha\beta}$, we have*

$$u\omega = \phi(u)\omega. \quad (12)$$

Proof 2.6 *Let $\omega = \phi(\omega)\eta_4^T$, $\eta_4 = [1, 0, 0, 0]$. We find by eq.[7] that*

$$u\omega = \phi(u\omega)\eta_4^T = \phi(u).\phi(\omega)\eta_4^T = \phi(u)\omega. \quad (13)$$

Definition 2.7 *A matrix X is called a quasi-orthogonal matrix if $X^T \varepsilon X = \varepsilon$*

$$\text{and } \det X = 1, \text{ here } \varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix} \text{ and } \alpha, \beta \in R.$$

3 Homothetic Exponential Motions With Generalized Quaternions

Definition 3.1 *Let $f(t) = e^t A$ and $B(t) = h(t)f(t)$ be the orthogonal matrix. Where $h(t) \neq \text{constant}$, $t \in R$.*

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ 1 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ 1 \end{bmatrix} \quad (15)$$

which is called homothetic exponential motion. The homothetic scale h the elements of f are continuously differentiable function of a real parameters t .

Where X and X_0 correspond to position vectors the same point with respect to the rectangular coordinate systems of the moving space R_0 and the fixed space R , At first time $t = t_0$ we consider coordinate systems of R_0 and R coincident. Then homothetic exponential motions of R_0 with respect to R will be denoted by $R_0 \setminus R$. $\forall t$,

Let $f' = Af$, and suppose $C \neq 0$ then

$$B' = \frac{dB}{dt} = h'f + hf' = (h' + hA)f. \quad (16)$$

where $h = h(t)$ is a scalar matrix, its inverse and transpose are

$$h^{-1} = \frac{1}{h} \cdot I, \quad h^T = h \quad (17)$$

respectively. Since f is an orthogonal matrix, the inverse of B is

$$B^{-1} = h^{-1}f^T, \quad f^{-1} = f^T \quad (18)$$

Theorem 3.2 *The matrix defined by the equation in eq.[20] is a quasi-orthogonal matrix.*

Proof 3.3 *Since $A^T \varepsilon A = \varepsilon$ and $\det A = 1$.*

Theorem 3.4 *The homothetic exponential motion given by eq.[16] is regular for all n and it is independent of h .*

Proof 3.5 $B' = h'f + hf' = (h' + hA)f = hf(A + \lambda I)$, where if we define $\lambda(t) = -\frac{h'(t)}{h(t)}$, then last equation is

$$B' = hf(A - \lambda I). \quad (19)$$

from eq. [21] we find that

$$\det B' = \det(hf) \det(A - \lambda I). \quad (20)$$

As $\det B' = 0$, that is, B' is singular, we get

$$h = 0 \text{ or } \det(A - \lambda I) = 0. \quad (21)$$

Here $h \neq 0$. Otherwise, the motion will be pure translation. B' is always regular.

4 POLE POINTS OF EXPONENTIAL MOTIONS

Definition 4.1 *To find the pole points, we have to solve the equation*

$$B'X_0 + C'. \quad (22)$$

Any solution of the equation in eq.[30] is a pole point of the motion at that instant in R_0 . The equation in eq.[30] has only one solution,

$$X_0 = -(B')^{-1}C \quad (23)$$

at every t -instant [7].

Theorem 4.2 *$f'(t)$ is a derivation of $f(t)$. then, the pole points corresponding to each t -instant in R_0 is the rotation by $(B)^{-1}$ of the speed vector C of the translation vector at the moment.*

Proof 4.3 *Since B' is quasi-orthogonal [4], then the matrix $(B')^{-1}$ is orthogonal. Thus it makes a rotation.*

Theorem 4.4 *If f is a orthogonal $n \times n$ matrix, the n^{th} -order derivatives of B are given by*

$$B^{(n)} = \left[\sum_{j=0}^n \binom{n}{j} h^{(n-j)} A^j \right] f. \quad (24)$$

Proof 4.5 *The proof of this theorem can be defined by induction. For $n=1$,*

$$B' = (h' + hA)f = \left[\sum_{j=0}^1 \binom{1}{j} h^{(1-j)} A^j \right] f. \quad (25)$$

Thus we have shown that it is true for $(n-1)$ and

$$B^{(n-1)} = \left[\sum_{j=0}^{n-1} \binom{n-1}{j} h^{(n-j-1)} A^j \right] f. \quad (26)$$

It can now be shown that it is true for n . Thus, for n :

$$B^{(n)} = [B^{(n-1)}]' = \left[\sum_{j=0}^{n-1} \binom{n-1}{j} h^{(n-j-1)} A^j \right] f. \quad (27)$$

By using the formulas the result can be given as follows:

$$B^{(n)} = \left[\sum_{j=0}^n \binom{n}{j} h^{(n-j)} A^j \right] f. \quad (28)$$

which proves the theorem.

Theorem 4.6 *In spaces of n -dimension the high order velocities of homothetic exponential motions are given by*

Proof 4.7 *The proof of this theorem can be defined by induction. For $n=1$,*

$$X' = BX_0 + BX'_0 + C' \text{ or } X' = \left[\sum_{j=0}^1 \binom{1}{j} B^{(1-j)} X_0^{(j)} + C' \right]. \quad (29)$$

Shown that it is true for $n = 1$ and $(n - 1)$. For n :

$$X^{(n)} = n0B^{(n)}X_0 + n1B^{(n-1)}X'_0 + \dots + nnBX_0^{(j)} + C^{(n)} \quad (30)$$

or

$$X^{(n)} = \sum_{j=0}^n \binom{n}{j} B^{(n)} X_0^{(j)} + C^{(n)}. \quad (31)$$

and we find

$$X^{(n)} = \sum_{j=0}^n \left[\sum_{i=0}^{n-j} \binom{n-j}{i} h^{(n-j-i)} A^i \right] f X_0^{(j)} + C^{(n)}. \quad (32)$$

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