On Generating Matrices of the $k$-Pell, $k$-Pell-Lucas
and Modified $k$-Pell Sequences

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Abstract

In this paper we define some tridiagonal matrices depending of a parameter from which we will find the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers.

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Keywords: $k$-Pell sequences; $k$-Pell-Lucas sequences; Modified $k$-Pell sequences; Recurrence Relations.

1 Introduction

The sequences of Pell, Pell-Lucas and Modified Pell numbers are sequences of numbers that are defined by the recursive recurrences. For $n$ a non-negative integer, the Pell sequence $\{P_n\}_n$, Pell-Lucas sequence $\{Q_n\}_n$ and Modified Pell sequence $\{q_n\}_n$ are given for $n \geq 2$, respectively, by the following recurrence relations with the respective initial conditions: $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$; $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = Q_1 = 2$; $q_n = 2q_{n-1} + q_{n-2}$, $q_0 = q_1 = 1$. More detail can be found in the extensive literature dedicated to these sequences. Still, we refer some examples of papers about some their properties: [2], [3], [1], [6], [13], [7], among others. More recently, P. Catarino [8], [11] and P. Catarino and P. Vasco [9], [10] did some research about the sequences of numbers

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1 Member of CM-UTAD (Research Centre of Mathematics of UTAD) and Collaborator of LabDCT-CIDTFF (Research Centre of “Didactics and Technology in Education of Trainers”- Pole of UTAD).
that arising from these sequences: for any positive real number $k$, the $k$-Pell sequence $\{P_{k,n}\}$, $k$-Pell-Lucas sequence $\{Q_{k,n}\}$ and Modified $k$-Pell sequence $\{q_{k,n}\}$, that are also defined by recursive recurrences. In these cases, for $n \geq 1$, we have, respectively, the following recurrence relations with the respective initial conditions: 

\[ P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad P_{k,0} = 0, \quad P_{k,1} = 1; \quad Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \quad Q_{k,0} = Q_{k,1} = 2; \quad q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, \quad q_{k,0} = q_{k,1} = 1. \]

The Binet’s formula for these types of numbers is, respectively, where $\sqrt{1+k}$ and $\sqrt{1-k}$ are the roots of the characteristic equation of the sequences $\{P_{k,n}\}$, $\{Q_{k,n}\}$ and $\{q_{k,n}\}$, respectively. As a curiosity, for $k = 1$, we obtain that $r_1$ is the silver ratio which is related with the Pell number sequence. Easily, from their Binet’s formula, we have that $2q_{k,j} = Q_{k,j}$, for all $j \geq 0$, one well-know relation between the terms of the $k$-Pell-Lucas and Modified $k$-Pell sequences.

From the definition of the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers, we present the first few values of the sequences in the following table:

<table>
<thead>
<tr>
<th>$j$</th>
<th>$P_{k,j}$</th>
<th>$Q_{k,j}$</th>
<th>$q_{k,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$2$</td>
<td>$2k + 4$</td>
<td>$k + 2$</td>
</tr>
<tr>
<td>$3$</td>
<td>$k + 4$</td>
<td>$6k + 8$</td>
<td>$3k + 4$</td>
</tr>
<tr>
<td>$4$</td>
<td>$4k + 8$</td>
<td>$2k^2 + 16k + 16$</td>
<td>$k^2 + 8k + 8$</td>
</tr>
<tr>
<td>$5$</td>
<td>$k^2 + 12k + 16$</td>
<td>$10k^2 + 40k + 32$</td>
<td>$5k^2 + 20k + 16$</td>
</tr>
<tr>
<td>$6$</td>
<td>$3k^2 + 28k + 32$</td>
<td>$2k^3 + 36k^2 + 96k + 64$</td>
<td>$k^3 + 18k^2 + 48k + 32$</td>
</tr>
<tr>
<td>$7$</td>
<td>$k^3 + 18k^2 + 72k + 64$</td>
<td>$14k^3 + 112k^2 + 224k + 128$</td>
<td>$7k^3 + 56k^2 + 112k + 64$</td>
</tr>
</tbody>
</table>

Table 1: The first eight $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers.

The purpose of this paper is to find the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers using some tridiagonal matrices. We follow closely some part of what S. Falcon did in the paper [12] for $k$-Fibonacci numbers.

2 The determinant of a special kind of tridiagonal matrices

In this section we use the matrices defined by A. Feng in [4] and applied to the
three types of numbers referred before and find the \( k \)-Pell, \( k \)-Pell-Lucas and Modified \( k \)-Pell numbers. We consider tridiagonal matrices in a similar way that Falcon did in [12]. In linear algebra a tridiagonal matrix is a matrix that has nonzero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. A tridiagonal matrix is a matrix that is both upper and lower Hessenberg matrix. Let us consider the square matrix of order \( \geq 1 \), denoted by \( M_n \), and defined (as in Falcon [12]) by

\[
M_n = \begin{pmatrix}
    a & b & 0 & 0 & 0 \\
    c & d & e & 0 & \cdots \\
    0 & c & d & e & 0 \\
    \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & 0 & \cdots e \\
    0 & 0 & 0 & 0 & \cdots e
\end{pmatrix},
\]

where \( a, b, c, d, e \) are real numbers. From some properties, the determinant of a tridiagonal matrix of order \( n \) can be computed from a recurrence relation. In this case, for each \( n \), if we compute the several determinants \( |M_n| \), we obtain that

\[
|M_1| = a \\
|M_2| = d|M_1| - bc \\
|M_3| = d|M_2| - ce|M_1| \\
|M_4| = d|M_3| - ce|M_2| \\
\vdots \\
|M_{n+1}| = d|M_n| - ce|M_{n-1}|.
\]

and, in general,

\[
|M_{n+1}| = d|M_n| - ce|M_{n-1}|. \tag{1}
\]

### 3 Some tridiagonal matrices and the \( k \)-Pell numbers

- If \( a = d = 2, b = e = k \) and \( c = -1 \), the matrix \( M_n \) above are transformed in the tridiagonal matrices,

\[
P_n(k) = \begin{pmatrix}
    2 & k & 0 & 0 & \cdots & 0 \\
    -1 & 2 & k & 0 & \cdots & 0 \\
    0 & -1 & 2 & k & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & 0 & \cdots & -1 & 2 \\
    0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix} \tag{2}
\]

In this case, and taking into account Table 1, the above formulas are transformed in

\[
|P_1(k)| = 2 = P_{k,2} \\
|P_2(k)| = 2|P_1(k)| + k = k + 4 = P_{k,3} \\
|P_3(k)| = 2|P_2(k)| + k|P_1(k)| = 4k + 8 = P_{k,4}
\]
\[ |P_4(k)| = 2|P_3(k)| + k|P_2(k)| = k^2 + 12k + 16 = P_{k,5} \]

and (1) is given by,
\[ |P_{n+1}(k)| = 2|P_n(k)| + k|P_{n-1}(k)|, \text{ for } n \geq 1. \]

Then we have the following result that gives us the \( k \)-Pell number of order \( n \) in terms of the determinant of a tridiagonal matrix:

**Proposition 1:** If \( P_n(k) \) is the \( n \)-by-\( n \) tridiagonal matrix considered in (2), then the \( n \)th \( k \)-Pell number is given by \( |P_{n-1}(k)| = P_{k,n} \). ■

• Also, using the tridiagonal matrix (2.3) considered in [5] for any second order linear recurrence sequence \( \{x_n\} \) such that \( x_{n+1} = Ax_n + Bx_{n-1}, n \geq 1 \), with \( x_0 = C, x_1 = D \). From the recurrence relation that define the \( k \)-Pell sequence and consider also the respective initial conditions, we consider \( C = 0, D = 2, A = 2, B = k \) and the correspondent tridiagonal \( n + 1 \)-by-\( n + 1 \) matrix is, in this case,

\[
P'_n(k) = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2 & k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 2 \\
0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]

(3)

and again taking into account Table 1, we obtain

\[
|P'_0(k)| = 0 = P_{k,0} \\
|P'_1(k)| = 1 = P_{k,1} \\
|P'_2(k)| = 2 = P_{k,2} \\
|P'_3(k)| = k + 4 = P_{k,3} \\
|P'_4(k)| = 4k + 8 = P_{k,4} \\
\vdots
\]

and then

**Proposition 2:** Let \( P'_n(k) \) be the \( n + 1 \)-by-\( n + 1 \) tridiagonal matrix considered in (3), then the \( n \)th \( k \)-Pell number is given by \( |P'_n(k)| = P_{k,n} \). ■

4 Some tridiagonal matrices and the \( k \)-Pell-Lucas numbers

• If \( a = 2k + 4, d = 2, b = 2k, e = k \) and \( c = -1 \), the matrix \( M_n \) above are transformed in the tridiagonal matrices,
On generating matrices

Taking into account Table 1, we have
\[
\begin{vmatrix}
2k + 4 & 2k & 0 & 0 & 0 \\
-1 & 2k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]
(4)

and (1) is given by,
\[
\begin{vmatrix}
2 & 2 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

Hence

**Proposition 3:** If \( Q_n(k) \) is the \( n \)-by-\( n \) tridiagonal matrix considered in (4), then the \( n \)th \( k \)-Pell-Lucas number is given by
\[
\begin{vmatrix}
2 & 2 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

\[\Box\]

- Again using the tridiagonal matrix (2.3) considered in [5] from the recurrence relation that define the \( k \)-Pell-Lucas sequence and consider also the respective initial conditions, we consider \( C = D = A = 2, B = k \) and the correspondent square tridiagonal \( n + 1 \)-by-\( n + 1 \) matrix is, in this case,

\[
\begin{vmatrix}
2 & 2 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

(5)

and we obtain
\[
\begin{vmatrix}
2 & 2 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

\[\Box\]

**Proposition 4:** Let \( Q'_n(k) \) be the \( n + 1 \)-by-\( n + 1 \) tridiagonal matrix considered in (5), then the \( n \)th \( k \)-Pell-Lucas number is given by
\[
\begin{vmatrix}
2 & 2 & 0 & 0 & \cdots & 0 \\
-1 & 0 & k & 0 & \cdots & 0 \\
0 & -1 & 2k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}
\]

\[\Box\]
5 Some tridiagonal matrices and the Modified \( k \)-Pell numbers

We know that, \( 2q_{k,j} = Q_{k,j} \) and so using the results established on Propositions 3 and 4 we can also find the term of order \( n \) of the Modified \( k \)-Pell sequence as a value of some determinant of some tridiagonal matrices. Hence we have

**Proposition 5:** If \( q_{k,n} \) are the \( n \)th Modified \( k \)-Pell number, then we have:

1. \( \frac{1}{2} |Q_{n}(k)| = q_{k,n} \);
2. \( \frac{1}{2} |Q_{n-1}(k)| = q_{k,n} \).

The tridiagonal matrices \( q_{n}(k) \) and \( q'_{n}(k) \) correspondent to the matrices (4) and (5) are, respectively

\[
q_{n}(k) = \begin{pmatrix}
    k + 2 & k & 0 & 0 & \cdots & 0 \\
    -1 & 2 & k & 0 & \cdots & 0 \\
    0 & -1 & 2 & k & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & -1 & 2 & k \\
    0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]

and

\[
q'_{n}(k) = \begin{pmatrix}
    1 & 1 & 0 & 0 & \cdots & 0 \\
    -1 & 0 & k & 0 & \cdots & 0 \\
    0 & -1 & 2 & k & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & -1 & 2 & k \\
    0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]

Some properties of the determinants of (square) matrices, allows us to find the same results of the Proposition 5.

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**References**

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