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# On Generating Matrices of the *k*-Pell, *k*-Pell-Lucas and Modified *k*-Pell Sequences

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**Abstract**

In this paper we define some tridiagonal matrices depending of a parameter from which we will find the *k*-Pell, *k*-Pell-Lucas and Modified *k*-Pell numbers.

**Mathematics Subject Classification:** 11B37, 11B83.

**Keywords**: *k*-Pell sequences; *k*-Pell-Lucas sequences; Modified *k*-Pell sequences; Recurrence Relations.

## 1 Introduction

The sequences of Pell, Pell-Lucas and Modified Pell numbers are sequences of numbers that are defined by the recursive recurrences. For $n$ a non-negative integer, the Pell sequence $\{P_n\}_n$, Pell-Lucas sequence $\{Q_n\}_n$ and Modified Pell sequence $\{q_n\}_n$ are given for $n \geq 2$, respectively, by the following recurrence relations with the respective initial conditions: $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ ; $Q_n = 2Q_{n-1} + Q_{n-2}$ , $Q_0 = Q_1 = 2$ ; $q_n = 2q_{n-1} + q_{n-2}$ , $q_0 = q_1 = 1$. More detail can be found in the extensive literature dedicated to these sequences. Still, we refer some examples of papers about some their properties: [2], [3], [1], [6], [13], [7], among others. More recently, P. Catarino [8], [11] and P. Catarino and P. Vasco [9], [10] did some research about the sequences of numbers

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that arising from these sequences: for any positive real number $k$, the $k$-Pell sequence $\{P_{k,n}\}_n$, $k$-Pell-Lucas sequence $\{Q_{k,n}\}_n$ and Modified $k$-Pell sequence $\{q_{k,n}\}_n$, that are also defined by recursive recurrences. In these cases, for $n \geq 1$, we have, respectively, the following recurrence relations with the respective initial conditions: $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$, $P_{k,0} = 0$, $P_{k,1} = 1$; $Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}$, $Q_{k,0} = Q_{k,1} = 2$; $q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}$, $q_{k,0} = q_{k,1} = 1$. The Binet's formula for these types of numbers is, $P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$, $Q_{k,n} = r_1^n + r_2^n$, $q_{k,n} = \frac{r_1^n + r_2^n}{r_1 + r_2} = \frac{r_1^n + r_2^n}{2}$, respectively, where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation of the sequences $\{P_{k,n}\}_n$, $\{Q_{k,n}\}_n$ and $\{q_{k,n}\}_n$, respectively. As a curiosity, for $k = 1$, we obtain that $r_1$ is the silver ratio which is related with the Pell number sequence. Easily, from their Binet's formula, we have that $2q_{k,j} = Q_{k,j}$, for all $j \geq 0$, one well-know relation between the terms of the $k$-Pell-Lucas and Modified $k$-Pell sequences.

From the definition of the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers, we present the first few values of the sequences in the following table:

| | $P_{k,j}$ | $Q_{k,j}$ | $q_{k,j}$ |
|---|---|---|---|
| $j = 0$ | $0$ | $2$ | $1$ |
| $j = 1$ | $1$ | $2$ | $1$ |
| $j = 2$ | $2$ | $2k + 4$ | $k + 2$ |
| $j = 3$ | $k + 4$ | $6k + 8$ | $3k + 4$ |
| $j = 4$ | $4k + 8$ | $2k^2 + 16k + 16$ | $k^2 + 8k + 8$ |
| $j = 5$ | $k^2 + 12k + 16$ | $10k^2 + 40k + 32$ | $5k^2 + 20k + 16$ |
| $j = 6$ | $3k^2 + 28k + 32$ | $2k^3 + 36k^2 + 96k + 64$ | $k^3 + 18k^2 + 48k + 32$ |
| $j = 7$ | $k^3 + 18k^2 + 72k + 64$ | $14k^3 + 112k^2 + 224k + 128$ | $7k^3 + 56k^2 + 112k + 64$ |

Table 1: The first eight $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers.

The purpose of this paper is to find the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers using some tridiagonal matrices

We follow closely some part of what S. Falcon did in the paper [12] for $k$-Fibonacci numbers.

## 2 The determinant of a special kind of tridiagonal matrices

In this section we use the matrices defined by A. Feng in [4] and applied to the three types of numbers referred before and find the $k$-Pell, $k$-Pell-Lucas and Modified $k$-Pell numbers. We consider tridiagonal matrices in a similar way that Falcon did in [12]. In linear algebra a tridiagonal matrix is a matrix that has nonzero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. A tridiagonal matrix is a matrix that is both upper and lower Hessenberg matrix. Let us consider the square matrix of order $\geq 1$, denoted by $M_n$, and defined (as in Falcon [12]) by

$$M_n = \begin{pmatrix} a & b & 0 & 0 & & & 0 \\ c & d & e & 0 & \cdots & & 0 \\ 0 & c & d & e & & & 0 \\ & \vdots & & & \vdots & & \vdots \\ & \ddots & & & \ddots & & \ddots \\ 0 & 0 & 0 & 0 & \dots & c \; d & e \\ 0 & 0 & 0 & 0 & \dots & 0 \; c & d \end{pmatrix},$$

where $a$, $b$, $c$, $d$ e $e$ are real numbers. From some properties, the determinant of a tridiagonal matrix of order $n$ can be computed from a recurrence relation. In this case, for each $n$, if we compute the several determinants $|M_n|$, we obtain that

$$\begin{aligned} |M_1| &= a \\ |M_2| &= d|M_1| - bc \\ |M_3| &= d|M_2| - ce|M_1| \\ |M_4| &= d|M_3| - ce|M_2| \\ &\vdots \end{aligned}$$

and, in general,

$$|M_{n+1}| = d|M_n| - ce|M_{n-1}|. \tag{1}$$

## 3 Some tridiagonal matrices and the $k$-Pell numbers

• If $a = d = 2$, $b = e = k$ and $c = -1$, the matrix $M_n$ above are transformed in the tridiagonal matrices,

$$P_n(k) = \begin{pmatrix} 2 & k & 0 & 0 & & & & 0 \\ -1 & 2 & k & 0 & & \cdots & & 0 \\ 0 & -1 & 2 & k & & & & 0 \\ & \vdots & & & & \vdots & & \vdots \\ & \ddots & & & & \ddots & & \ddots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & k \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \tag{2}$$

In this case, and taking into account Table 1, the above formulas are transformed in

$$\begin{aligned} |P_1(k)| &= 2 = P_{k,2} \\ |P_2(k)| &= 2|P_1(k)| + k = k + 4 = P_{k,3} \\ |P_3(k)| &= 2|P_2(k)| + k|P_1(k)| = 4k + 8 = P_{k,4} \end{aligned}$$

$$|P_4(k)| = 2|P_3(k)| + k|P_2(k)| = k^2 + 12k + 16 = P_{k,5}$$
$$\vdots$$

and (1) is given by,

$$|P_{n+1}(k)| = 2|P_n(k)| + k|P_{n-1}(k)|, \text{ for } n \geq 1.$$

Then we have the following result that gives us the $k$-Pell number of order $n$ in terms of the determinant of a tridiagonal matrix:

**Proposition 1:** If $P_n(k)$ is the $n$-by-$n$ tridiagonal matrix considered in (2), then the $n^{\text{th}}$ $k$-Pell number is given by $|P_{n-1}(k)| = P_{k,n}$. ■

• Also, using the tridiagonal matrix (2.3) considered in [5] for any second order linear recurrence sequence $\{x_n\}$ such that $x_{n+1} = Ax_n + Bx_{n-1}, n \geq 1$, with $x_0 = C, x_1 = D$. From the recurrence relation that define the $k$-Pell sequence and consider also the respective initial conditions, we consider $C = 0, D = 2, A = 2, B = k$ and the correspondent tridiagonal $n+1$-by-$n+1$ matrix is, in this case,

$$P'_n(k) = \begin{pmatrix} 0 & 1 & 0 & 0 & & & & 0 \\ -1 & 0 & k & 0 & \cdots & & & 0 \\ 0 & -1 & 2 & k & & & & 0 \\ & \vdots & & & & \vdots & & \vdots \\ & \ddots & & & & \ddots & & \ddots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & k \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} \tag{3}$$

and again taking into account Table 1, we obtain

$$|P'_0(k)| = 0 = P_{k,0}$$
$$|P'_1(k)| = 1 = P_{k,1}$$
$$|P'_2(k)| = 2 = P_{k,2}$$
$$|P'_3(k)| = k + 4 = P_{k,3}$$
$$|P'_4(k)| = 4k + 8 = P_{k,4}$$
$$\vdots$$

and then

**Proposition 2:** Let $P'_n(k)$ be the $n+1$-by-$n+1$ tridiagonal matrix considered in (3), then the $n^{\text{th}}$ $k$-Pell number is given by $|P'_n(k)| = P_{k,n}$. ■

## 4 Some tridiagonal matrices and the $k$-Pell-Lucas numbers

• If $a = 2k + 4$, $d = 2$, $b = 2k$, $e = k$ and $c = -1$, the matrix $M_n$ above are transformed in the tridiagonal matrices,

$$Q_n(k) = \begin{pmatrix} 2k+4 & 2k & 0 & 0 & & & 0 \\ -1 & 2 & k & 0 & & \cdots & 0 \\ 0 & -1 & 2 & k & & & 0 \\ & \vdots & & & & \vdots & \vdots \\ & \ddots & & & & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \dots & -1 \quad 2 & k \\ 0 & 0 & 0 & 0 & \dots & 0 \quad -1 & 2 \end{pmatrix} \tag{4}$$

Taking into account Table 1, we have

$$|Q_1(k)| = 2k+4 = Q_{k,2}$$
$$|Q_2(k)| = 2|Q_1(k)| + 2k = 6k+8 = Q_{k,3}$$
$$|Q_3(k)| = 2|Q_2(k)| + k|Q_1(k)| = 2k^2+16k+16 = Q_{k,4}$$
$$|Q_4(k)| = 2|Q_3(k)| + k|Q_2(k)| = 10k^2+40k+32 = Q_{k,5}$$
$$\vdots$$

and (1) is given by,

$$|Q_{n+1}(k)| = 2|Q_n(k)| + k|Q_{n-1}(k)|, \text{ for } n \geq 1.$$

Hence

**Proposition 3:** If $Q_n(k)$ is the $n$-by-$n$ tridiagonal matrix considered in (4), then the $n^{\text{th}}$ $k$-Pell-Lucas number is given by $|Q_{n-1}(k)| = Q_{k,n}$. ■

• Again using the tridiagonal matrix (2.3) considered in [5] from the recurrence relation that define the $k$-Pell-Lucas sequence and consider also the respective initial conditions, we consider $C = D = A = 2, B = k$ and the correspondent square tridiagonal $n+1$-by-$n+1$ matrix is, in this case,

$$Q'_n(k) = \begin{pmatrix} 2 & 2 & 0 & 0 & & & 0 \\ -1 & 0 & k & 0 & & \cdots & 0 \\ 0 & -1 & 2 & k & & & 0 \\ & \vdots & & & & \vdots & \vdots \\ & \ddots & & & & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \dots & -1 \quad 2 & k \\ 0 & 0 & 0 & 0 & \dots & 0 \quad -1 & 2 \end{pmatrix} \tag{5}$$

and we obtain

$$|Q'_0(k)| = 2 = Q_{k,0}$$
$$|Q'_1(k)| = 2 = Q_{k,1}$$
$$|Q'_2(k)| = 2k+4 = Q_{k,2}$$
$$|Q'_3(k)| = 6k+8 = Q_{k,3}$$
$$\vdots$$

and then

**Proposition 4:** Let $Q'_n(k)$ be the $n+1$-by-$n+1$ tridiagonal matrix considered in (5), then the $n^{\text{th}}$ $k$-Pell-Lucas number is given by $|Q'_n(k)| = Q_{k,n}$. ■

## 5 Some tridiagonal matrices and the Modified *k*-Pell numbers

We know that, $2q_{k,j} = Q_{k,j}$ and so using the results established on Propositions 3 and 4 we can also find the term of order $n$ of the Modified $k$-Pell sequence as a value of some determinant of some tridiagonal matrices. Hence we have

**Proposition 5:** If $q_{k,n}$ are the $n^{\text{th}}$ Modified $k$-Pell number, then we have:

1. $\frac{1}{2}|Q'_n(k)| = q_{k,n}$;
2. $\frac{1}{2}|Q_{n-1}(k)| = q_{k,n}$. ■

The tridiagonal matrices $q_n(k)$ and $q'_n(k)$ correspondent to the matrices (4) and (5) are, respectively

$$q_n(k) = \begin{pmatrix} k+2 & k & 0 & 0 & & & & 0 \\ -1 & 2 & k & 0 & & \cdots & & 0 \\ 0 & -1 & 2 & k & & & & 0 \\ & \vdots & & & & \vdots & & \vdots \\ & \ddots & & & & \ddots & & \ddots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & k \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

and

$$q'_n(k) = \begin{pmatrix} 1 & 1 & 0 & 0 & & & & 0 \\ -1 & 0 & k & 0 & & \cdots & & 0 \\ 0 & -1 & 2 & k & & & & 0 \\ & \vdots & & & & \vdots & & \vdots \\ & \ddots & & & & \ddots & & \ddots \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & k \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}.$$

Some properties of the determinants of (square) matrices, allows us to find the same results of the Proposition 5.

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