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Some Maximum Modulus Polynomial Rings and Constant Modulus Spaces

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Abstract

Let $F \in C^1(\Omega, \mathbb{C})$ be a not necessarily open function on a Euclidean manifold Ω such that F obeys the strong maximum modulus principle in a bilateral sense defined in the paper and does not attain weak local minimum on a submanifold $M \subset \Omega$. We prove that the polynomial ring $\mathbb{C}[F]$ satisfies the strong maximum modulus principle on M . We also give a sufficient condition for subspaces of polyanalytic functions to have constant modulus spaces containing only constants.

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1 Introduction

The paper consists of two parts. In the first part of this paper (Section 2) we provide polynomial rings generated by not necessarily open functions which obey the strong maximum modulus principle such that this principle is preserved. The second part (Section 3) concerns constant modulus families and conditions for reduction to constants. The maximum principle referred to in the first part is the following.

Definition 1.1 Let M, K be real or complex Euclidean manifolds. A family of continuous functions $\mathcal{A} \subset C(M, K)$ is said to obey the *strong maximum modulus principle* if whenever $f \in \mathcal{A}$ attains weak local maximum it holds true

that f is constant. By a weak local maximum at a point $p_0 \in M$, we mean that there exists a domain $U \subset M$, $p_0 \in U$, such that $\max_{z \in \bar{U}} |f(z)| \leq |f(p_0)|$, and for simplicity we denote by $|\cdot|$ the norm in both M and K . When $K = \mathbb{R}$ we replace $|f|$ by f .

Our first main result is given in Section 2.1 and concern cases of not necessarily open functions $F \in C^1(M, K)$ where M, K are real or complex manifolds (spaces generated by open functions are more tractable, see Observation 2.1). In particular we give sufficient conditions on F such that for $K = \mathbb{C}$ the polynomial ring $\mathbb{C}[F]$ satisfies the strong maximum modulus principle (see Theorem 2.7).

In the second part (Section 3) we consider functions which have composition with generalizations of the norm, which are globally constant and conditions for rings where each member must be constant. In connection to this we also give a sufficient condition for subspaces of polyanalytic functions to have constant modulus spaces containing only constants, see Proposition 3.5.

1.1 Preliminaries on Section 3

Let M be a Euclidean manifold, $K = \mathbb{R}^n$ or $\mathbb{C}^n$, and let $f \in C(M, K)$. If f is an open map $M \rightarrow K$ and $U \subset M$ is open then $f(p), p \in U$, cannot yield a weak local maximum unless f is constant, because $f(U)$ must be an open neighborhood of $f(p)$. Hence openness of any map between Euclidean manifolds implies the strong maximum modulus principle. Real analytic functions are in general not open maps as opposed to nonconstant complex analytic functions. If $K = \mathbb{C}$ then $f(U)$ is connected for any connected $U \subset M$ (see Rudin [8], p.93) but not necessarily open. It is not necessary for a function $f \in C^1(M, K)$, which obeys the strong maximum modulus principle to be open.

Example 1.2 For $M = K = \mathbb{R}$ simple examples such as the harmonic function $f(x) = x_1^2$ mapping $(-1, 1)$ to the nonopen $[0, 1)$ give the answer. (for higher dimension one can use that separately harmonic on $\mathbb{R}^n$ implies harmonic). For $K = \mathbb{C}$ take e.g. the map $f := x_1^2 + ix_2^2, \mathbb{R}^2 \rightarrow \mathbb{C}$, which is nonopen but the function $|f|^2 = x_1^4 + x_2^4$ satisfies $\Delta |f|^2 = 4 \cdot 3(x_1^2 + x_2^2) > 0$, for $(x_1, x_2) \neq (0, 0)$, and $f(0, 0) = 0$, thus f does not attain weak local maximum at any point.

1.1.1 A transversality condition

For basics on transversality¹ see e.g. Hirsch [6].

¹We shall use the notation $\bar{\cap}$ for transversal intersection.

Observation 1.3 Let M, K be real or complex Euclidean manifolds and let $f \in C^1(M, K)$. Assume that $\forall p \in M$ there is a real C^1 -curve $\gamma \subset M$ passing p such that $f|_\gamma$ (which is necessarily connected because of continuity, i.e. it is either the point $\{f(p)\}$ or a curve in K) satisfies $f(\gamma) \bar{\cap} T_{f(p)}\partial B(|f(p)|)$ where $\partial B(|f(p)|)$ denotes the sphere in K . Then f obeys the strong maximum modulus principle.

Of course it is the transversality condition which is the main obstacle.

Example 1.4 Consider first the perhaps simplest case, namely M a (real or complex) Euclidean manifold and $K = \mathbb{R}$. Because by continuity the restriction of f to any real curve $p \in \gamma \subset M$ has connected image, we have that either f is constant along that curve or the image contains an open interval $I \subset K = \mathbb{R}$. If $f|_\gamma : \gamma \rightarrow \mathbb{R}$ is open then clearly $f(p) \in I$, so f obeys the strong maximum modulus principle. The condition that $f(\gamma) \bar{\cap} \partial B(|f(p)|)$ here simply reduces to (because $\partial B(|f(p)|)$ is simply the end points of an interval) $T_{f(p)}f(\gamma) + \{0\} = T_p\mathbb{R}$ which is fulfilled as soon as $T_{f(p)}f(\gamma) \neq \{0\}$, i.e. if we parametrize $\gamma = \gamma(t)$, $t \in (-1, 1)$, $\gamma(0) = p$, then it is sufficient that $(f \circ \gamma)'(0) \neq 0$ for f to be such that the image of any open neighborhood of p contains an open neighborhood of $f(p)$.

Remark 1.5 The conditions of Observation 1.3 are not necessary for the strong maximum principle to hold. Namely if there exists $p \in M$ through which there passes no submanifold γ along which the image of f intersects $\partial B_0(|f(p)|)$ transversally then the image of f must attain either a weak local maximum or a weak local minimum at p , so Observation 1.3 gives an equivalence to satisfying the strong maximum modulus principle only for maps which never attain either weak local maximum nor weak local minimum unless they are constant (e.g. linear maps).

Sometimes we shall use the term strictly increasing to one side along a curve γ passing p and by this we mean that γ is parametrized according to $(-1, 1) \ni t \mapsto z(t)$ and $|f(z(t))|$ is strictly increasing on $t \in [0, 1)$. Note that given $p \in M$, if there does not exist any such curve then the function must attain weak local maximum (since the modulus would necessarily be non-increasing in every direction) thus reduce to a constant by the strong maximum modulus principle.

2 Maximum modulus polynomial rings

We begin with an observation.

Observation 2.1 Let M be a Euclidean manifold. For any map $F \in C(M, \mathbb{C}^n)$ satisfying that for every $p \in M$ and every open $U \ni p$ there exists an open $V \subset F(U)$ containing $F(p)$, its composition, $H \circ F$ with any open self-map map H satisfies the strong maximum modulus principle. Namely a consequence of the fact that a set in $\mathbb{C}^n$ is open iff it is finitely open, is that we have that for any a map $F = (F_1, \dots, F_n) \in C(M, \mathbb{C}^n)$ satisfying that for every $p \in M$ and every open $U \ni p$ there exists an open $V \subset F(U)$ containing $F(p)$, it holds true that $F_j : M \rightarrow \mathbb{C}$ also has the same property, i.e. for every $p \in M$ and every open $U \ni p$ there exists an open $V_j \subset F_j(U) \subset \mathbb{C}$ containing $F_j(p)$. First of all $|F(p)|$ cannot be a weak local maximum for F at any p since on any neighborhood U the image of some point has larger modulus than $F(p)$ in the open neighborhood V of $F(p)$. The image under an open map H of $F(U)$ is again open ($H \circ F$ is open). Now F satisfies separately that for every $p \in M$ and every open $U \ni p$ there exists an open $V_j \subset F_j(U)$ containing $F_j(p)$. Then $(H(F))_j$ also satisfies this property so $|(H(F))_j|^2$ does not attain weak local maximum unless $(H(F))_j$ is constant thus also $|H(F)|^2 = \sum_{j=1}^n |(H(F))_j|^2$ satisfies the strong maximum modulus principle.

Question: Let M, K be Euclidean manifolds. Does there exists interesting polynomial rings over the field $\mathbb{C}$ which obey the strong maximum modulus principle and contain a non-open map?

Since such a ring² containing one non-open map f automatically contains at least all of $\mathbb{C}[f]$ our primary concern is this smallest necessary case.

2.1 The not necessarily open case

Let M be a real or complex Euclidean manifold, $K = \mathbb{C}^n$ and define the componentwise product $f \cdot g := (f_1 \cdot g_1, \dots, f_n \cdot g_n)$, $f, g \in C(M, K)$. We

shall denote $f^k := \overbrace{f \cdots f}^{k\text{-times}}$. This binary operator is componentwise $\mathbb{C}$ -linear. We can thus turn the $\mathbb{C}$ -vector space $C(M, K)$ into a $\mathbb{C}$ -algebra by defining multiplication by a complex scalar $\mu \in \mathbb{C}$ according to $\mu f := \overbrace{(\mu, \dots, \mu)}^{n\text{-times}} \cdot f$, $f \in C(M, K)$. We shall further denote $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Observation 2.2 Let M be a real or complex manifold, and let $f \in C(M, \mathbb{C})$. $\lambda \cdot f^k$, $\lambda \in \mathbb{C}$ obeys the strong maximum modulus principle whenever f does (in the sense that it does not attain weak local maximum unless it is

²In this text we shall use the term polynomial ring a generalized sense, i.e. $R[f]$ is a ring consisting of the set of polynomials over f with coefficients in another ring R , i.e. R is not required to be a field. In the case of $R = \mathbb{C}$ this makes no difference but we can speak of polynomial rings over $\mathbb{C}^n$ or over $\mathcal{O}(M)$ (the set of holomorphic maps).

constant). To see this note that if f is identically zero or a nonzero constant the statements trivial so assume f to be nonconstant, thus does not attain weak local maximum. Then for a fix complex number $\lambda \in \mathbb{C}$ we have that $\lambda \cdot f^k$ increases in modulus whenever f^k does. For f^2 the statement follows from the fact that $|f^2(z(t))| = |f(z(t))|^2$. The last equality is a consequence of the fact that for $z \in \mathbb{C}$, $|z^2| = |z|^2$. Iteration yields the result for $|f^k|$ where k is any finite positive integer, namely $|\lambda f^k|^2 = |\lambda|^2 |f^k|^2 = |\lambda|^2 |f|^{2k}$, and since $|f|$ does not attain weak local maximum there exists a real curve $\gamma_p \subset M$ passing p , say given by a parametrization $t \mapsto z(t) \in M, t \in (-1, 1), z(0) = p$. such that $|f(z(t))|$ strictly increases to one side of $t = 0$. This means that also every factor of $|f(z(t))|^{2k}$ increases to the same side of p along γ_p thus $|\lambda|^2 |f(p)|^{2k}$ cannot be a weak local maximum on M .

The statement of Observation 2.2 breaks down if one tries to generalize to the cases $n > 1$. Already in the case of real valued λ this can easily be seen.

Example 2.3 Let $f \in C(\mathbb{R}^2, \mathbb{R}^2)$ be defined by,

$$(x_1, x_2) \mapsto \left(1 - \sqrt{x_1^2 + x_2^2}, 1 + \sqrt{x_1^2 + x_2^2}\right), \quad (1)$$

which certainly obeys the strong maximum modulus principle since,

$$|f|^2 = 2 + 2(x_1^2 + x_2^2) \quad (2)$$

never attains weak local maximum. However for $\lambda = (\lambda_1, \lambda_2) = (2, 1) \in \mathbb{R}^2$ we have,

$$|\lambda \cdot f|^2 = 4(1 - 2|x| + |x|^2) + (1 + 2|x| + |x|^2) = 5 - 6|x| + 5|x|^2, \quad (3)$$

which attains strict local maximum at the origin $|f(0, 0)|^2 = 5$.

More affirmative results can be obtained using the transversality condition.

Definition 2.4 Let M, K be real or complex manifolds both of real dimension ≥ 2 . A function $f \in C^1(M, K)$ is said to satisfy the strong maximum modulus principle *bilaterally* if for every point p , either $f(p)$ is a local minimum or there exists two local C^1 -curves $\gamma_{p,1}, \gamma_{p,2}$ passing p such that $f(\gamma_{p,1}), f(\gamma_{p,2})$ (which are necessarily connected) are C^1 -curves satisfying $f(\gamma_{p,1}) \overline{\cap}_{f(p)} f(\gamma_{p,2})$ and $f(\gamma_{p,j}) \overline{\cap}_{f(p)} \partial B_0(|f(p)|)$, $j = 1, 2$.

We automatically have that such a function f satisfies the usual strong maximum modulus principle, indeed by Remark 1.5 it is sufficient that $f(\gamma_{p,j}) \overline{\cap}_{f(p)} \partial B_0(|f(p)|)$ for at least one $j = 1, 2$. For real dimension ≥ 2 note that two C^1 -curves (of nonconstant modulus in K , i.e. not lying on a sphere)

passing a point p on the unit sphere can have transversal intersection at p but both belong to $\overline{B_0(F(p))}$.³

Example 2.5 Clearly any open map $f \in C^1(M, K)$ for M, K real or complex manifolds both of real dimension ≥ 2 , satisfies the strong maximum modulus principle bilaterally. This includes all (nonconstant) holomorphic functions when M, K are complex manifolds, and we pointed out in the introduction that the space of open nonholomorphic functions is rich.

The following example shows that the phenomenon of bilaterally increasing also occurs without the assumption of C^1 -smoothness but we shall in this text only use the notion of strong maximum modulus principle bilaterally with respect to the C^1 -smooth case.

Example 2.6 Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (x_1, x_2, \sin[x_3])$, where $[x_3]$ denotes the least upper integer. We have $|f|^2 = x_1^2 + x_2^2 + \sin^2[x_3]$. At a fix point (p_1, p_2, p_3) there are two curves $\gamma_{p,1} := \{(t, p_2, p_3), t \in (-\epsilon_p, \epsilon_p)\}$, $\gamma_{p,2} := \{(p_1, t, p_3), t \in (-\epsilon'_p, \epsilon'_p)\}$ along which $|f|$ strictly increases and such that their image under f yield two transversal curves in the target space thus f satisfies the strong maximum principle and always at least in two directions however f is not continuous.

2.1.1 Statement and proof of the main theorem

Theorem 2.7 *Let Ω be a real or complex smooth manifold and let $F \in C^1(\Omega, \mathbb{C})$ satisfy the strong maximum modulus principle bilaterally and assume it does not attain weak local minimum on a submanifold $M \subset \Omega$. Then the polynomial ring $\mathbb{C}[F]$ satisfies the strong maximum modulus principle on M .*

Proof. Consider the translation $G = cF^k + c'$ where $c' \in \mathbb{C}$. Since F obeys the strong maximum modulus principle bilaterally we have given $p \in M$ either $|F(p)|$ is a weak local minimum or there exists two C^1 -curves $\gamma_{p,1}, \gamma_{p,2}$ passing p along which $|F|$ increases (in the same direction as F) and their images under F are C^1 -curves which intersect transversally at $F(p)$, and therefore also $|cF^k| = |c||F|^k$ (recall $n = 1$) increases (in the same direction as $|F|$ does) along both of $\gamma_{p,1}, \gamma_{p,2}$ through p . By assumption p is not a weak local minimum for $|F|$.

Lemma 2.8 *Any polynomial of the form $P(F)(z) = c' + cF^k(z)$ $c', c \in \mathbb{C}$, not only satisfies the strong maximum modulus principle but given any $p \in M$, $|P(F)|$ increases (in the same direction as $|F|$ does) along at least one of $\gamma_{p,j}, j = 1, 2$ and $(P(F))(\gamma_{p,1}) \overline{\cap}_{(P(F))(p)} (P(F))(\gamma_{p,2})$.*

³A natural generalization in real dimension $n > 2$ would be the concept of being n -laterally increasing, i.e. with the requirement of n pairwise transversal C^1 -curves $f(\gamma_{p,j})$, such that $f(\gamma_{p,j}) \overline{\cap}_{f(p)} f(\gamma_{p,k}), j \neq k$ and $\sum_{j=1}^n T_{f(p)} f(\gamma_{p,j}) = T_{f(p)} K$, but for our purposes it will be sufficient to consider the bilateral case.

Figure 1: Rotations will not disrupt transversality to the spheres involved of one curve whereas translation by a complex number in general will. But for the pair $F(\gamma_{p,j}), j = 1, 2$ translation will preserve the property of increasing (in modulus) along at least one of the two curves $F(\gamma_{p,j}), j = 1, 2$ (here we are assuming that we first pick two $\gamma_{p,j}$ in case there are more choices of pairs of curves satisfying the condition). For simplicity we have drawn the two curves with almost orthogonal intersection, but it is sufficient that they intersect transversally.

Proof. Let $X^{(1)} = (x_1^{(1)} + ix_2^{(1)})$ ($X_2 = x_1^{(2)} + ix_2^{(2)}$) be a vector in $\mathbb{C}$ such that $\mathbb{R}X^{(1)}$ ($\mathbb{R}X^{(2)}$) is the tangent space at $F(p)$ of the local real curve $F(\gamma_{p,1})$ ($F(\gamma_{p,2})$). Now the tangent space of the translation $c' + F(\gamma_{p,j})$ at the point $c' + F(p)$ is also spanned by $X^{(j)} = (x_1^{(j)} + ix_2^{(j)})$, $j = 1, 2$ thus the translations of the local curves intersect transversally at $c' + F(p)$ (see Figure 1). This means that for any $c'' \in \mathbb{C}, c'' \neq 0$ we have (because multiplication by a complex constant in $\mathbb{C}$ simply amounts to a rotation followed by a scaling, thus preserves transversality of two C^1 -curves) $c' + cF = c \left(\frac{c'}{c} + F \right)$, which is a complex multiple of a translation of F satisfies that,

$$(c' + cF(\gamma_{p,1})) \overline{\cap}_{F(p)} (c' + cF(\gamma_{p,2})). \quad (4)$$

Because of this transversality $X^{(j)}, j = 1, 2$ cannot both belong to the tangent space of the sphere $\partial B_0(|c' + cF(p)|)$ at $c' + cF(p)$ because they each have complementary dimension to the sphere (note that this is only true for $n = 1$ in higher dimension one would need to introduce the notion of being n -lateral instead of bilateral, to obtain an analogue statement).

We now affirm the preservation of the transversality condition satisfied by F , for powers F^k , i.e. that $F(\gamma_{p,j}) \nparallel_{F(p)} \partial B_0(|F^k(p)|)$ $j = 1, 2$ and $F^k(\gamma_{p,1}) \nparallel_{F^k(p)} F^k(\gamma_{p,2})$. To this end we first note that F is differentiable and the images under F^k of the local curves $\gamma_{p,1}, \gamma_{p,2}$ are locally near $F^k(p)$ connected, see e.g. Rudin [8], p.93, thus themselves C^1 -curves through and sufficiently near $F^k(p)$ since $|F|$ increases along them both⁴ thus so does $|F^k| = |F|^k$ (when $K = \mathbb{C}$), so it indeed holds true that,

$$F^k(\gamma_{p,j}) \nparallel_{F^k(p)} \partial B_0(|F^k(p)|), j = 1, 2. \quad (5)$$

Let us parametrize $F(\gamma_{p,j}), j = 1, 2$ (sufficiently near $F(p)$ such that $F^k(\gamma_{p,j}), j = 1, 2$ can also be assumed local C^1 -curves) according to,

$$\begin{cases} F(\gamma_{p,1}) : (-1, 1) \ni t \mapsto F(\alpha(t)) = \operatorname{Re}F(\alpha(t)) + i\operatorname{Im}F(\alpha(t)), \alpha(0) = p, \\ F(\gamma_{p,2}) : (-1, 1) \ni s \mapsto F(\beta(t)) = \operatorname{Re}F(\beta(t)) + i\operatorname{Im}F(\beta(t)), \beta(0) = p, \end{cases} \quad (6)$$

where α, β are $C^1((-1, 1), M)$. We shall assume M has real dimension L , i.e. $\alpha(t) = (\alpha_1(t), \dots, \alpha_L(t)), \beta(t) = (\beta_1(t), \dots, \beta_L(t))$. We have

$$x_1^{(1)} = (\nabla \operatorname{Re}F|_p) \cdot \left(\frac{\partial}{\partial t} \alpha(t) \right) |_{t=0} = \begin{bmatrix} \frac{\partial \operatorname{Re}F}{\partial x_1}|_p & \dots & \frac{\partial \operatorname{Re}F}{\partial x_L}|_p \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha_1(t)}{\partial t}|_{t=0} \\ \vdots \\ \frac{\partial \alpha_L(t)}{\partial t}|_{t=0} \end{bmatrix} \quad (7)$$

and similarly,

$$\begin{aligned} x_2^{(1)} &= \begin{bmatrix} \frac{\partial \operatorname{Im}F}{\partial x_1}|_p & \dots & \frac{\partial \operatorname{Im}F}{\partial x_L}|_p \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha_1(t)}{\partial t}|_{t=0} \\ \vdots \\ \frac{\partial \alpha_L(t)}{\partial t}|_{t=0} \end{bmatrix}, \\ x_1^{(2)} &= \begin{bmatrix} \frac{\partial \operatorname{Re}F}{\partial x_1}|_p & \dots & \frac{\partial \operatorname{Re}F}{\partial x_L}|_p \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_1(t)}{\partial t}|_{t=0} \\ \vdots \\ \frac{\partial \beta_L(t)}{\partial t}|_{t=0} \end{bmatrix}, \\ x_2^{(2)} &= \begin{bmatrix} \frac{\partial \operatorname{Im}F}{\partial x_1}|_p & \dots & \frac{\partial \operatorname{Im}F}{\partial x_L}|_p \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_1(t)}{\partial t}|_{t=0} \\ \vdots \\ \frac{\partial \beta_L(t)}{\partial t}|_{t=0} \end{bmatrix}. \end{aligned} \quad (8)$$

⁴This is one of the reasons we do not merely require $F \in C(M, K)$ namely we need C^1 -curves in K .

Now consider F^k and let $Y^{(1)} = (y_1^{(1)} + iy_2^{(1)})$ ($Y_2 = y_1^{(2)} + iy_2^{(2)}$) be a vector in $\mathbb{C}$ such that $\mathbb{R}Y^{(1)}$ ($\mathbb{R}Y^{(2)}$) is the tangent space at $F^k(p)$ of the local real curve $(F^k)(\gamma_{p,1})$ ($(F^k)(\gamma_{p,2})$). We have that,

$$\begin{aligned} y_1^{(1)} &= (\nabla \operatorname{Re}(F^k)|_p) \cdot \left(\frac{\partial}{\partial t} \alpha(t) \right) |_{t=0} = \left[\frac{\partial \operatorname{Re}(F^k)}{\partial x_1} |_p \quad \dots \quad \frac{\partial \operatorname{Re}(F^k)}{\partial x_L} |_p \right] \begin{bmatrix} \frac{\partial \alpha_1(t)}{\partial t} |_{t=0} \\ \vdots \\ \frac{\partial \alpha_L(t)}{\partial t} |_{t=0} \end{bmatrix}, \\ y_2^{(1)} &= \left[\frac{\partial \operatorname{Im}(F^k)}{\partial x_1} |_p \quad \dots \quad \frac{\partial \operatorname{Im}(F^k)}{\partial x_L} |_p \right] \begin{bmatrix} \frac{\partial \alpha_1(t)}{\partial t} |_{t=0} \\ \vdots \\ \frac{\partial \alpha_L(t)}{\partial t} |_{t=0} \end{bmatrix}, \end{aligned} \quad (9)$$

and similarly for $y_1^{(2)}, y_2^{(2)}$. We now verify that $\nabla(F^k)(p)$ ($= \nabla(\operatorname{Re}(F^k)(p)) + i\nabla(\operatorname{Im}(F^k)(p))$) $= (kF^{k-1}(p))\nabla F(p)$ which in turn implies (since $F(p) = 0$ would imply a local minimum so we can assume $F(p) \neq 0$) that $(kF^{k-1}(p))X^{(j)} = Y^{(j)}$ thus $Y^{(1)}, Y^{(2)}$ are not real multiples of each other as soon as $X^{(1)}, X^{(2)}$ are not real multiples of each other. This implies that $F^k(\gamma_{p,1}) \nparallel_{F^k(p)} F^k(\gamma_{p,2})$ since we then know that the modulus of F^k increases (in the same direction as F) along $\gamma_{p,j}, j = 1, 2$. For the sake of completeness we prove the aforementioned (perhaps easy) relation. Let us first consider the simple case $k = 2$. We have $\operatorname{Re}(F^2) = (\operatorname{Re}F)^2 - (\operatorname{Im}F)^2$, $\operatorname{Im}(F^2) = 2(\operatorname{Re}F)(\operatorname{Im}F)$, thus,

$$\frac{\partial \operatorname{Re}(F^2)}{\partial x_i} |_p = 2\operatorname{Re}F(p) \frac{\partial \operatorname{Re}F}{\partial x_i} |_p - 2\operatorname{Im}F(p) \frac{\partial \operatorname{Im}F}{\partial x_i} |_p, i = 1, \dots, L. \quad (10)$$

$$\frac{\partial \operatorname{Im}(F^2)}{\partial x_i} |_p = 2\operatorname{Re}F(p) \frac{\partial \operatorname{Im}F}{\partial x_i} |_p + 2\operatorname{Im}F(p) \frac{\partial \operatorname{Re}F}{\partial x_i} |_p, i = 1, \dots, L, \quad (11)$$

hence the i :th component of $\nabla(F^2)|_p$ is,

$$\begin{aligned} (\nabla(F^2)|_p)_i &= (\nabla(\operatorname{Re}(F^2))|_p + i\nabla(\operatorname{Im}(F^2))|_p)_i = \\ &= 2\operatorname{Re}F(p) \frac{\partial \operatorname{Re}F}{\partial x_i} |_p - 2\operatorname{Im}F(p) \frac{\partial \operatorname{Im}F}{\partial x_i} |_p + i \left(2\operatorname{Re}F(p) \frac{\partial \operatorname{Im}F}{\partial x_i} |_p + 2\operatorname{Im}F(p) \frac{\partial \operatorname{Re}F}{\partial x_i} |_p \right) = \\ &= 2 \left(\underbrace{\operatorname{Re}F(p) + i\operatorname{Im}F(p)}_{=F(p)} \right) \left(\frac{\partial \operatorname{Re}F}{\partial x_i} + i \frac{\partial \operatorname{Im}F}{\partial x_i} \right) |_p, 1 \leq i \leq L. \end{aligned} \quad (12)$$

This takes care of the case $k = 2$. Next we use induction in $k \geq 2$. Assume $(\nabla(F^{k-1})|_p)_j = (k-1)F^{k-2}(p)(\nabla F|_p)_j$. Note that,

$$\begin{aligned} (\nabla F^k|_p)_j &= (\nabla(F^{k-1} \cdot \operatorname{Re} F)|_p)_j + i(\nabla(F^{k-1} \cdot \operatorname{Im} F)|_p)_j = \\ &= (\operatorname{Re} F \underbrace{\nabla(F^{k-1})|_p}_{=((k-1)F^{k-2}(p)\nabla F|_p)_j})_j + (F^{k-1}(p) \cdot (\nabla \operatorname{Re} F|_p))_j + i(\operatorname{Im} F(p)\nabla(F^{k-1})|_p)_j + \\ &= (F^{k-1}(p) \cdot (\nabla \operatorname{Im} F|_p))_j = (\operatorname{Re} F(p) + i\operatorname{Im} F(p))((k-1)F^{k-2}(p)\nabla F|_p)_j + \\ &= F^{k-1}(p)((\nabla \operatorname{Re} F|_p)_j + i(\nabla \operatorname{Im} F|_p)_j) = ((k-1)F^{k-1}(p)\nabla F|_p)_j + \\ &= (F^{k-1}(p)\nabla F|_p)_j = (kF^{k-1}(p)\nabla F|_p)_j. \end{aligned} \quad (13)$$

This completes the induction. Thus we conclude that if $Y^{(1)}$ and $Y^{(2)}$ are spanning tangent vectors at $F^k(p)$ for the curves $F^k(\gamma_{p,1})$ and $F^k(\gamma_{p,2})$ respectively it holds true that they can only be real multiples of each other when $X^{(1)}$ and $X^{(2)}$ are real multiples of each other, which is excluded, hence we have,

$$F^k(\gamma_{p,1}) \nmid_{F^k(p)} F^k(\gamma_{p,2}), 1 \leq k < \infty. \quad (14)$$

Now repeating the same arguments for translation, but with F replaced by $cF^k, c \in \mathbb{C}, k$ -a finite integer, we obtain that

$$(c' + cF^k(\gamma_{p,j})) \nmid_{(c'+cF^k(p))} \partial B_0(|c' + cF^k(p)|), \text{ for at least one } j = 1, 2. \quad (15)$$

This proves Lemma 2.8. $\square$

To complete the proof we shall now use induction in $N \geq 2$. Assume that for any sum of $N-1$ monomials $(Q(F))(z) = \sum_{j=1}^{N-1} a_j F^{k_j}(z), a_j \in \mathbb{C}$, with $k_j, 1 \leq j \leq N-1$ positive integers, it holds true that Q increases along at least one of $\gamma_{p,j}, j = 1, 2$ and $(Q(F))(\gamma_{p,1})$ has transversality intersection with $(Q(F))(\gamma_{p,2})$ at $(Q(F))(p)$ (for the case $N = 2$ this follows from Lemma 2.8). Let $d_N \in \mathbb{C}^*$, let k_N be a positive integer, assume w.l.o.g. that $k_N \geq k_{N-1} \geq \dots \geq k_1 \geq 0$, and set $P(F) = \sum_{j=1}^N d_j F^{k_j}, d_j \in \mathbb{C}$. We can rewrite this as,

$$P(F) = d_1 F^{k_1} \underbrace{\left(1 + \underbrace{\sum_{j=2}^N \frac{d_j}{d_1} F^{k_j - k_1}}_{=: P''(F)} \right)}_{=: P'(F)}. \quad (16)$$

By what we have already done we know that $d_1 F^{k_1}$ increases along both $\gamma_{p,j}, j = 1, 2$ and by the induction hypothesis we have (because $P''(F)$ is the sum of $N-1$ monomials in F) that $|P''|$ not only increases along at least one

of $\gamma_{p,j}$, $j = 1, 2$, say $\gamma_{p,1}$, but also satisfies that $(P''(F))(\gamma_{p,2})$ has transversal intersection with $(P''(F))(\gamma_{p,1})$ at $(P''(F))(p)$. Hence by (i) it holds true that P' , which is a translation of P'' , must increase (in the same direction as $|F|$) along at least one of $\gamma_{p,j}$, $j = 1, 2$, thus $|P(F)|$ cannot attain weak local maximum at p . By induction no polynomial in F with coefficients in $\mathbb{C}$ can attain weak local maximum at p . This completes the proof. $\square$

Corollary 2.9 *If $F \in C(\Omega, \mathbb{C}^n)$ and there exists $1 \leq l \leq n$ such that $F_l \in C(\Omega, \mathbb{C})$ satisfies the strong maximum modulus principle bilaterally and does not attain weak local minimum on a submanifold $M \subset \Omega$, and $F_j \in \mathbb{C}[F_l]$, $j \neq l$. Then $\mathbb{C}^n[F]$ satisfies the strong maximum modulus principle on M .*

Proof. Since F_l satisfies the strong maximum modulus principle Theorem 2.7 applies to F_j , $1 \leq j \leq n$, to yield that $\mathbb{C}[F_j]$ satisfies the the strong maximum modulus principle. Now every member $P(F) \in \mathbb{C}^n[F]$, say $P(F) = c_1 F + \cdots + c_N F^N$, where $c_j = (c_{j1}, \dots, c_{jN})$, is separately a polynomial, namely $(P(F))_j = c_{j1} F_j + \cdots + c_{jN} F_j^N$. Furthermore ever F_j is a polynomial $F_j = d_{j1} F_l + \cdots + d_{jL} F_l^{L_j}$. Thus $(P(F))_j$ is itself a polynomial of degree $N \cdot L$ in F_l . By the proof of Theorem 2.7 (i.e. the case $n = 1$) it holds true not only that $(P(F))_j : M \rightarrow \mathbb{C}$ obeys the strong maximum modulus principle, but further if it is not constant, that there exists through each $p \in M$ a curve γ_p such that $|F_l|$ and $|(P(F))_j|$ both increase to at least one (and the same) side of p along γ_p (note that the same l can be used independent of $1 \leq j \leq n$). If it is constant we must have that F_l is constant thus also F is constant in which case we are done since polynomials over a constant are again constant thus obey the strong maximum modulus principle. So we can assume F_l is nonconstant. Hence every term of $|P(F)|^2 = \sum_{j=1}^n |(P(F))_j|^2$, is nonconstant (since every time one was constant that forced F_l to be constant) and increases (to the same side of p as F_l) along γ_p , thus the sum is strictly increasing to one side of p along γ_p , thus $P(F)$ cannot attain weak local maximum at p unless it is constant. $\square$

Observation 2.10 Let M, K be real or complex Euclidean manifolds, let $\Omega \Subset M$ be a domain and denote by $\mathcal{M} \subset C^1(M, K)$ the subspace of functions which satisfy the strong maximum modulus principle bilaterally and do not attain weak local maximum on Ω . Any function $f : M \rightarrow K$ which can be uniformly approximated on a neighborhood⁵ of $\overline{\Omega}$ by a family $\{P_j\}_{j \in \mathbb{N}}$, $P_j \in \mathcal{M}$, satisfies on any subdomain $\omega \subseteq \Omega$, $\max_{z \in \overline{\omega}} |f(z)| = \max_{z \in \partial \omega} |f(z)|$. The proof of this observation is quite simple so we give it as an appendix.

⁵Note that f automatically becomes continuous on a neighborhood of $\overline{\Omega}$ since it is a uniform limit of continuous functions.

3 Constant modulus spaces

3.1 Polyanalytic functions

A higher order generalization of holomorphic functions in one variable are the so called polyanalytic functions. For preliminaries on polyanalytic of order q (q -analytic) functions see the survey article Balk [3] and the references therein

Definition 3.1 (q -analytic function) Let $0 \in U \subset \mathbb{C}$ be a domain. A function f on U of order n is called q -analytic (or *polyanalytic of order q*) at 0 if it has the form,

$$f(z) = \sum_{j=0}^{q-1} a_j(z) \bar{z}^j, a_j \in \mathcal{O}(U). \quad (17)$$

If $a_{q-1} \not\equiv 0$ then q is called the *exact* order. $f(z) \in C^q(U)$ is q -analytic on U (where q is a positive finite integer) iff,

$$\frac{\partial^q f}{\partial \bar{z}^q} = 0 \text{ on } U. \quad (18)$$

(see Balk [3], p.198).

The following characterization is known due to Balk [2].

Theorem 3.2 (Balk [2]) *A polyanalytic function of order n in a domain $\Omega \subset \mathbb{C}$ has a constant modulus iff f is representable in the form $f(z) = \lambda \cdot \overline{P(z)}/P(z)$ where $P(z)$ is a polynomial of degree not higher than $n-1$, and λ is a constant.*

In the holomorphic (i.e. 1-analytic) case the reduction of constant modulus spaces to spaces of constants is a consequence of the combination of the two conditions,

$$(i) \quad \frac{\partial}{\partial \bar{z}} f(z) = 0, z \in \Omega \quad (ii) \quad |f| \equiv \text{constant on } \Omega. \quad (19)$$

When the first condition is generalized to $\frac{\partial^q}{\partial \bar{z}^q} f(z) = 0$ it is natural to ask for sufficient strengthening of the second condition for preserving the reduction of constant modulus spaces to spaces of constants.

Observation 3.3 If $\Omega \subset \mathbb{C}$ is a domain and $f \in C^q(\Omega)$ then,

$$(i) \quad \frac{\partial^q}{\partial \bar{z}^q} f(z) = 0, z \in \Omega \quad (ii) \quad \left| \frac{\partial^j f}{\partial \bar{z}^j} \right| \equiv \text{constant on } \Omega, \quad 0 \leq j < q-1, \quad (20)$$

implies that $f \equiv \text{constant on } \Omega$.

Proof. This is a consequence of the case $q = 1$. It is sufficient to prove the result near a point $p \in \Omega$ and w.l.o.g. we assume $p = 0 \in \Omega$. By definition we know that f can be represented on an open $U \subset \Omega$ as $f(z) = \sum_{j=0}^{q-1} a_j(z) \bar{z}^j$, $a_j \in \mathcal{O}(U)$. On U we have,

$$\frac{\partial^{q-1}}{\partial \bar{z}^{q-1}} f(z) \equiv a_{q-1}(z), \quad (21)$$

thus the condition (ii) of Eqn.(20) implies that the holomorphic function $a_{q-1}(z)$ has constant modulus on U and therefore $a_{q-1} \equiv \text{constant}$. Iteration of this process yields that each $a_j \equiv \text{constant}$, $0 \leq j \leq q-1$, thus $f \equiv \text{constant}$ on U . Furthermore we can repeat this for each $p \in \Omega$ i.e. find an open U_p on which $f \equiv \text{constant}$. Since Ω is relatively compact there is a finite cover consisting of open sets on which f is constant. Since Ω is a domain f reduces to a global constant on Ω . $\square$

Note that the representation $\overline{P(z)}/P(z)$, for a complex polynomial P only defines a polyanalytic function on a domain which must depend upon the zeros of P .

Example 3.4 Let $f(z) = \frac{\bar{z} - \bar{p}_1}{z - p_1}$, for a fix $p_1 \in \mathbb{C}$. Clearly $|f(z)| \equiv 1$ and $P(z)$ is a holomorphic polynomial of degree one but $f(z)$ does not define a polyanalytic function at p_1 . Recall that any polyanalytic function is continuous (it is a finite sum of finite products of holomorphic and antiholomorphic functions) Assume for simplicity $p_1 = 0$, let $z := x + iy$ and rewrite $f(z) = \frac{\bar{z}^2}{|z|^2} = 1 - \frac{y(2y + i2x)}{x^2 + y^2}$. Approaching the origin along the line $x(t) = t, y(t) = 0$ we have $\lim_{t \rightarrow 0} f(x(t) + iy(t)) = 0$, whereas approach along the line $x(t) = t, y(t) = t$ we have $\lim_{t \rightarrow 0} f(x(t) + iy(t)) = 1 - \lim_{t \rightarrow 0} \frac{t^2(2+2i)}{2t^2} = -i$, thus $f(z)$ is not continuous at the origin.

Given the characterization of Theorem 3.2 we may relax the condition (ii) of Eqn.(20).

Proposition 3.5 *Let $\Omega \subset \mathbb{C}$ be a domain. Any function $f(z)$ of constant modulus on Ω satisfying,*

$$(i) \quad \frac{\partial^q}{\partial \bar{z}^q} f(z) = 0, z \in \Omega \quad (ii) \quad \left| \frac{\partial f}{\partial \bar{z}} \right| \equiv \text{constant on } \Omega, \quad (22)$$

must reduce to a constant on Ω .

Proof. Let $f(z) = \overline{P(z)}/P(z)$, for a polynomial P of degree at most $q-1$. If we find a sufficient condition for any such f to reduce to a constant then obviously any $\lambda \cdot f, \lambda \in \mathbb{C}$ is also a constant thus Theorem 3.2 will imply that such a condition will be sufficient for the reduction of constant modulus spaces to constants. Let the degree of P be $0 \leq r \leq q-1$. If $r = 0$ we are done since

then f is then holomorphic and of constant modulus thus a constant. So let $1 \leq r \leq q-1$. By the fundamental theorem of algebra we can for some points $p_1, \dots, p_r$ write,

$$f(z) = \frac{(\bar{z} - \bar{p}_1) \cdots (\bar{z} - \bar{p}_r)}{(z - p_1) \cdots (z - p_r)}. \quad (23)$$

Since Ω does not contain any zeros of $P(z)$, f is complex differentiable on Ω and we can calculate,

$$\frac{\partial}{\partial \bar{z}} f(z) = \sum_{j=1}^r \alpha_j \cdot \frac{1}{(z - p_j)}, \quad \alpha_j := \prod_{\substack{1 \leq k \leq r \\ j \neq k}} \left(\frac{\bar{z} - \bar{p}_k}{z - p_k} \right). \quad (24)$$

Assume in order to arrive at a contradiction that $\left| \frac{\partial}{\partial \bar{z}} f \right| \equiv C_1$ on Ω , for a real constant C_1 . Since,

$$\alpha_j(z) \cdot \frac{z - p_j}{\bar{z} - \bar{p}_j} = f(z). \quad (25)$$

we obtain (recall that Ω does not contain any zeros of $P(z)$),

$$C_1 \equiv \left| \sum_{j=1}^r \frac{1}{\bar{z} - \bar{p}_j} \right| = \left| \sum_{j=1}^r \frac{1}{z - p_j} \right|, \forall z \in \Omega. \quad (26)$$

Since the sum in the right hand side defines a holomorphic function on Ω this implies that $\sum_{j=1}^r \frac{1}{z - p_j} \equiv \text{constant}$ on the open subset $\Omega \subset \mathbb{C}$, which is impossible. $\square$

3.2 Rings with canonical fix or constant modulus spaces

Let $\mathcal{A} \subset C(\Omega, \mathbb{C})$ for an open $\Omega \subseteq \mathbb{C}$. For a nonnegative constant c denote the constant modulus space $[c]_{\mathcal{A}} := \{g \in \mathcal{A} : |g(z)| = c, \forall z \in \Omega\}$. It is natural to ask when $[c]_{\mathcal{A}}$ contains only constant functions. We already know that $\mathcal{A} = \mathcal{O}(\Omega)$ is sufficient for this property.

Question: Is $\mathcal{R} = \mathcal{O}(\Omega)$ a maximal ring in $C^\infty(\Omega, \mathbb{C})$ with the property that $\forall c \geq 0$, the constant modulus sets $([c]_{\mathcal{R}})$ consist only of constant functions?

We can give a partially affirmative answer.

Proposition 3.6 *$\mathcal{O}(\Omega)$ has no ring extension in $C^\infty(\Omega)$ containing a non-holomorphic function of constant modulus, whose constant modulus sets consist only of constant functions.*

Proof. Let $f \in C^\infty(\Omega, \mathbb{C}) \setminus \mathcal{O}(\Omega)$ and let $\mathcal{A}$ be the ring extension $\mathcal{O}(\Omega)[f]$ i.e. every element of $\mathcal{A}$ is of the form $\sum_{k=1}^m g_k \cdot f^k + h$ for some holomorphic g_k, h and

integer $0 \geq m < \infty$. Assume in order to arrive at a contradiction: $\forall c \geq 0$, $[c]_{\mathcal{A}}$ consists only of constant functions, and that there is a nonholomorphic function of constant modulus. In particular $|\sum_{k=1}^m g_k \cdot f^k + h| \equiv \text{constant}$ implies that $\sum_{k=1}^m g_k \cdot f^k$ is holomorphic so,

$$\left(\sum_{k=1}^m k g_k f^{k-1} \right) \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \Omega. \quad (27)$$

Since f is not holomorphic on Ω we can find an open subset $\mathfrak{U}_1 \subset \Omega$ on which (by continuity) $\frac{\partial f}{\partial \bar{z}}$ is nowhere zero. This requires $\sum_{k=1}^m k g_k f^{k-1} \equiv 0$ on $\mathfrak{U}_1$. Hence,

$$\left(\sum_{k=2}^m k(k-1) g_k f^{k-2} \right) \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \mathfrak{U}_1. \quad (28)$$

As before there must exist an open subset $\mathfrak{U}_2 \subset \mathfrak{U}_1$ such that, $\frac{\partial f}{\partial \bar{z}}$ is nowhere zero on $\mathfrak{U}_2$ thus $\sum_{k=2}^m k(k-1) g_k f^{k-2} \equiv 0$ on $\mathfrak{U}_2$. Repeating this process we obtain finally an open subset $\mathfrak{U}_{m-1} \subset \mathfrak{U}_1 \subset \Omega$ such that,

$$m! g_m \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \mathfrak{U}_{m-1}. \quad (29)$$

Since g_m has only isolated zeros there is a dense open subset $\omega \subset \mathfrak{U}_{m-1}$ such that, $\frac{\partial f}{\partial \bar{z}} \equiv 0$ on ω . By continuity of $\frac{\partial f}{\partial \bar{z}}$ on Ω this implies that $f \in \mathcal{O}(\mathfrak{U}_{m-1})$ which is a contradiction to the choice of $\mathfrak{U}_1$. This completes the proof. $\square$

3.2.1 Fix modulus spaces

A generalization of constant modulus function spaces on a Euclidean manifold Ω , are fix modulus spaces, by which we mean families of the form $\{g \in C(\Omega, \mathbb{C}) : |g(z)| \equiv |f(z)|, \forall z \in \Omega\}$, for fix $f \in C(\Omega, \mathbb{C})$. Let $\mathcal{A} \subset C(\Omega, \mathbb{C})$ be a family of functions and denote $[f]_{\mathcal{A}} := \{g \in \mathcal{A} : |g(z)| \equiv |f(z)|, \forall z \in \Omega\}$.⁶ Clearly $[f]_{\mathcal{A}}$ contains $\{\lambda f : \lambda \in \mathbb{C}, |\lambda| = 1\}$ as soon as $\mathbb{C}\mathcal{A} = \mathcal{A}$. If $[1]_{\mathcal{A}}$ consists only of constant functions and if $g \in \mathcal{A}$ is zero free on Ω then $|f| \equiv |g| \Rightarrow \left| \frac{f}{g} \right| \equiv 1 \Rightarrow f \equiv |g| e^{it_0}$ for some $t_0 \in [0, 2\pi)$. For holomorphic functions $f, g \in \mathcal{O}(\Omega)$, $|f| \equiv |g|$ implies $f \equiv e^{it} g$, for some $t \in [0, 2\pi)$, in fact a more general result is known due to Debiard & Gaveau [4]: Let $D \subset \mathbb{C}^n$ be a bounded domain and let $f, g \in \mathcal{O}(D) \cap C(\bar{D})$. If $|f| \equiv |g|$ on ∂D then $f \equiv e^{it} g$ for some $t \in [0, 2\pi)$. We also mention that Globenvik [5] proved a result which implies that for every $\varphi : \partial D$ such that $\log \varphi$ is integrable defines the norm $\|f\| : \partial D \rightarrow \mathbb{R}_{\geq 0}$, for some analytic disc⁷ $f : \bar{D} \rightarrow \mathbb{C}^2$, (further related results

⁶For $f(z)$ a nonnegative constant c independent of z , $[f]_{\mathcal{A}}$ reduces to a constant modulus space $[c]_{\mathcal{A}} = \{g \in \mathcal{A} : |g(z)| = c, \forall z \in \Omega\}$.

⁷By analytic disc we mean that the map f belongs to $C(\bar{D})$ and holomorphic on D where D denotes the unit disc in $\mathbb{C}$.

on determining conditions for a nonnegative function can be the modulus of the trace of certain holomorphic functions is due to Shikorov [9] and Jacewicz [7].)

Question: Let $\Omega \subset \mathbb{C}$ be an open subset. Is $\mathcal{R} = \mathcal{O}(\Omega)$ a maximal ring in $C^\infty(\Omega, \mathbb{C})$ with the property that $[\phi]_{\mathcal{R}} \subset \{\psi \in \mathcal{R} : \psi = \lambda \cdot \phi + \mu, \lambda, \mu \in \mathbb{C}\}$ for all $\phi \in \mathcal{R}$?

Proposition 3.7 $\mathcal{O}(\Omega)$ has no ring extension $\mathcal{A}$ in $C^\infty(\Omega)$, containing a nonholomorphic function of constant modulus, such that the extension has the property $[f]_{\mathcal{A}} \subset \{g \in \mathcal{A} : g = \lambda \cdot f + \mu, \lambda, \mu \in \mathbb{C}\}$ for all $f \in \mathcal{A}$.

Proof. Let $f \in C^\infty(\Omega, \mathbb{C}) \setminus \mathcal{O}(\Omega)$ and let $\mathcal{A}$ be the ring extension $\mathcal{A} := \mathcal{O}(\Omega)[f]$. Assume in order to arrive at a contradiction: $[\phi]_{\mathcal{A}} \subset \{\psi \in \mathcal{A} : \psi = \lambda \cdot \phi + \mu, \lambda, \mu \in \mathbb{C}\}$ for all $\phi \in \mathcal{A}$. Let $G, H \in \mathcal{A}$ such that $|G| \equiv |H|$. We know that $G = \sum_{k=1}^m g_k \cdot f^k + h$ and $H = \sum_{k=1}^{m'} g'_k \cdot f^k + h'$, for some holomorphic g_k, h, g'_k, h' and positive integers m, m' . By assumption $|G| \equiv |H|$ implies that there exists $\lambda, \mu \in \mathbb{C}$ such that $G = \lambda H + \mu$. This gives,

$$\left(\sum_{k=1}^{m'} (g_k - \lambda g'_k) \cdot f^k \right) + \left(\sum_{k=m'+1}^m g_k \cdot f^k \right) + (h - h') = \mu, \quad m > m', \quad (30)$$

$$\left(\sum_{k=1}^m (g_k - \lambda g'_k) \cdot f^k \right) - \left(\sum_{k=m+1}^{m'} g'_k \cdot f^k \right) + (h - h') = \mu, \quad m \leq m'. \quad (31)$$

Applying $\frac{\partial}{\partial \bar{z}}$ to Eqn.(30), gives,

$$\left(\sum_{k=1}^{m'} k(g_k - \lambda g'_k) f^{k-1} + \sum_{k=m'+1}^m g_k \cdot f^{k-1} \right) \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \Omega. \quad (32)$$

Since f is not holomorphic on Ω we can find an open subset $\mathfrak{U}_1 \subset \Omega$ on which (by continuity) $\frac{\partial f}{\partial \bar{z}}$ is nowhere zero. This requires,

$$\left(\sum_{k=1}^{m'} k(g_k - \lambda g'_k) f^{k-1} + \sum_{k=m'+1}^m k g_k \cdot f^{k-1} \right) \equiv 0, \text{ on } \mathfrak{U}_1. \quad (33)$$

Repeating this process $m-1$ -times we obtain nonempty open subsets $\mathfrak{U}_{m-1} \subset \cdots \subset \mathfrak{U}_1 \subset \Omega$ such that,

$$m! g_m \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \mathfrak{U}_{m-1}. \quad (34)$$

Since g_m has only isolated zeros there is a dense open subset $\omega \subset \mathfrak{U}_{m-1}$ such that, $\frac{\partial f}{\partial \bar{z}} \equiv 0$ on ω which in turn implies that $f \in \mathcal{O}(\mathfrak{U}_{m-1})$ giving a contradiction. The case $m' \geq m$ is handled similarly but where Eqn.(34) is replaced by,

$$-m'!g_{m'} \cdot \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ on } \mathfrak{U}_{m'-1}, \quad (35)$$

where $g_{m'}$ has only isolated zeros. This completes the proof. $\square$

Note that Proposition 3.7 does not say that $\mathcal{O}(\Omega)$ is maximal with respect to for the property of interest, only that ring extensions in $C^\infty(\Omega)$ cannot have nonholomorphic functions which need to be checked for verifying the property.

Example 3.8 Let $\Omega \Subset \mathbb{C}$ be an open relatively compact subset and let $f(z) := e^{i\operatorname{Re} z}$, $z \in \Omega$. Clearly $\frac{\partial}{\partial \bar{z}} f(z) = \frac{if(z)}{2} \neq 0$, $\forall z \in \Omega$, so $f \in C^\infty(\Omega) \setminus \mathcal{O}(\Omega)$. The ring extension $(\mathcal{O}(\Omega))[f]$ certainly contains nonholomorphic functions (e.g. λf , $\lambda \in \mathbb{C} \setminus \{0\}$) of constant modulus which are also nonconstant. In particular $|\lambda f| = |\lambda|$, i.e. $\lambda f \in [\lambda]_{\mathcal{A}}$.

We mention that there are results on modulus of boundary values of holomorphic functions due to Shikorov [9], Jacewicz [7]. Shikorov [9] gives sufficient conditions for for a positive function on the unit circle in $\mathbb{C}$ to be the modulus a some Λ_ω^n -function (we do not repeat the definition here).

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A Proof of Observation 2.10

We shall use the following lemma.

Lemma A.1 *For any sufficiently small $r > 0$, there exists a relatively compact subdomain $\Omega(r) \Subset \Omega$ (in particular $\partial\Omega(r) \subset \Omega$) such that,*

$$\sup_{z \in \partial\Omega} \text{dist}(z, \partial\Omega(r)) < r. \quad (36)$$

Proof. We can cover $\partial\Omega$ by $\bigcup_{z \in \partial\Omega} B_z(r/2)$ (where $B_z(r/2)$ denotes the ball centered at p of radius $r/2$). Since Ω is relatively compact, $\partial\Omega$ is a compact thus there is a finite subcover $\bigcup_{1 \leq j \leq N} B_{z_j}(r/2)$ for a finite point set $\{z_j \in \partial\Omega\}_{1 \leq j \leq N}$, $N < \infty$. Setting,

$$\Omega(r) := \Omega \cap \left(\bigcup_{z_j \in \partial\Omega} (B_{z_j}(r/2))^c \right) \quad (37)$$

(where X^c denotes the complement in M of a subset X) we obtain that for sufficiently small $r > 0$, $\Omega(r) \subset \Omega$ will be a nonempty domain (necessarily relatively compact since Ω is). Furthermore we have for any $z \in \partial\Omega$, that $z \in B_{z_j}(r/2)$ for some $1 \leq j \leq N$, thus $\text{dist}(z, \partial\Omega(r)) < r$. This proves Lemma A.1. $\square$

We begin by proving the result for $\Omega \Subset M$ itself, and then it will be clear that the proof can be repeated for every subdomain $\omega \subseteq \Omega$. We need to prove,

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|. \quad (38)$$

By Theorem 2.7 any P_j obeys the strong maximum modulus principle on a neighborhood of $\overline{\Omega}$, thus (because the strong maximum modulus principle implies the boundary maximum modulus principle),

$$\max_{z \in \partial\Omega} |P_j(z)| = \max_{z \in \overline{\Omega}} |P_j(z)|. \quad (39)$$

Set $\max_{z \in \bar{\Omega}} |f| = c$ and let $\Pi(\Omega, f) := \{z \in \bar{\Omega} : |f(z)| = c\}$. If the proposition does not hold then there exists relatively compact subdomains $\Omega(r) \Subset \Omega$ (in particular we can assume $\partial\Omega(r) \subset \Omega$) depending on $0 < r$ sufficiently small such that the closed peak set $\Pi(\Omega, f)$ will not intersect the boundary $\partial\Omega$. So given sufficiently small $\epsilon > 0$ we can for sufficiently small $r > 0$, assume $\Pi(\Omega, f) \subset \Omega(r)$ and it can further be assumed that,

$$\max_{z \in \overline{\Omega(r)}} |f(z)| - 2\epsilon \geq \max_{z \in \partial\Omega} |f(z)|. \quad (40)$$

Now by uniform convergence on a neighborhood of $\bar{\Omega}$ there exists a finite integer $L \in \mathbb{N}$ such that,

$$\max_{z \in \bar{\Omega}} |P_j(z) - f(z)| < \epsilon/3, \forall j \geq L. \quad (41)$$

Also we have for $r > 0$ that $\max_{z \in \partial\Omega} |P_j(z)| = \max_{z \in \bar{\Omega}} |P_j(z)| \geq \max_{z \in \overline{\Omega(r)}} |P_j(z)|$. This gives,

$$\begin{aligned} \max_{z \in \partial\Omega} |f(z)| &\geq \max_{z \in \partial\Omega} |P_j(z)| - \epsilon/3 \geq \\ &\max_{z \in \overline{\Omega(r)}} |P_j(z)| - \epsilon/3 \geq \max_{z \in \overline{\Omega(r)}} |f(z)| - 2\epsilon/3, \quad \forall j \geq L. \end{aligned} \quad (42)$$

Eqn.(42) yields a contradiction to Eqn.(40), so we have proved Eqn.(38) which shows that the result holds true for $\omega = \Omega$. Now we can replace Ω in the above proof by any subdomain $\omega \subset \Omega$ (note that ω is again relatively compact in M since Ω is). We have: (i) P_j converge uniformly on a neighborhood of $\bar{\omega}$ and (ii) Lemma A.1 applies to ω . Thus each step of the proof can be repeated in the same fashion to yield $\omega(r')$, for sufficiently small $r' > 0$, to replace the role of $\Omega(r)$ above, thus allowing us to deduce,

$$\max_{z \in \bar{\omega}} |f(z)| = \max_{z \in \partial\omega} |f(z)|. \quad (43)$$

Since $\omega \subseteq \Omega$ was an arbitrary subdomain we are done. $\square$

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