An Integral Characterization for the Curves on the Hyperbolic Sphere H_0^3 in the Minkowski Space-time E_1^4

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Abstract

In this paper, we showed that a hyperbolic spherical curve in the Minkowski space-time E_1^4 can be given by Frenet-like equations of a space-like curve in the Minkowski 3-space E_1^3 . Thus, finding an integral characterization for a hyperbolic spherical E_1^4 -curve is identical to finding it for a space-like E_1^3 -curve. For a unit speed space-like curve $\alpha(s)$ in E_1^3 with $\alpha'(s) = T$, we showed that $\alpha(s)$ is a space-like Frenet curve with curvature $\kappa(s)$ and torsion $\tau(s)$ if and only if there are constant vectors a and b such that

$$T'(s) = \kappa(s) \left\{ ae^{\xi} + be^{-\xi} - \int_0^s \cosh(\xi(s) - \xi(\delta)) T(\delta) \kappa(\delta) d\delta \right\},$$

where $\xi(s) = \int_0^s \tau(\delta) d\delta$.

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1 Introduction

In the Euclidean 3-space E^3 , the differential equation characterizing a spherical curve is given by

$$\frac{d}{ds} \left\{ \frac{1}{\tau(s)} \frac{d\rho}{ds} \right\} + \rho(s) \tau(s) = 0, \quad (1)$$

where s is the arc length, $\rho(s) = 1/\kappa(s)$ is the radius of curvature and $\tau(s)$ is the torsion of the curve [5]. Note that, the equation (1) has a meaning if $\tau(s) \neq 0$ and $\kappa(s) \neq 0$. In [10], Wong gave an explicit solution of this differential equation as

$$\rho(s) = a \cos \left[\int_0^s \tau(t) dt \right] + b \sin \left[\int_0^s \tau(t) dt \right],$$

where a and b are arbitrary constants (see also [1]). On the other hand, Wong [9] gave a condition for a curve to be a spherical curve without any preconditions on $\tau(s)$ and $\kappa(s)$: A unit speed curve in the Euclidean 3-space E^3 is a spherical curve if and only if the curvature $\kappa(s)$ is nowhere zero (then its torsion $\tau(s)$ is defined), and there exists a function $f \in C^1$ such that

$$f\tau = (1/\kappa)', \quad f' + \tau/\kappa = 0.$$

Putting $\rho = 1/\kappa$ gives $\rho' = \tau f$, $f' = -\tau\rho$ which are the Frenet analogue equations for a unit-speed curve in the Euclidean plane E^2 . Similarly, Dannon [2] showed that spherical curves in the Euclidean 4-space E^4 can be given by Frenet analogue equations in E^3 : Let $\alpha(s)$ be an E^4 unit speed C^5 Frenet curve with curvature functions $\kappa(s)$, $\tau(s)$, $\mu(s)$. Then $\alpha(s)$ lies on an E^4 sphere if and only if $\kappa(s) \neq 0$ and there are two C^2 functions $f(s)$ and $g(s)$ so that

$$\rho' = \tau f, \quad f' = -\tau\rho + \mu g, \quad g' = -\mu f, \quad (\rho = 1/\kappa).$$

These equations are the Frenet analogue equations for a unit-speed curve in the Euclidean 3-space E^3 . Then he obtained an integral characterization for a spherical E^4 curve by finding it for an E^3 Frenet curve: Let $\alpha(s)$ be a C^4 curve in E^3 parameterized by its arc length s so that $\alpha'(s) = T$. Then $\alpha(s)$ has a Frenet system T, N, B , curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy the Frenet

equations $T'(s) = \kappa N$, $N'(s) = -\kappa T + \tau B$, $B'(s) = -\tau N$ if and only if there exist constant vectors a and b so that

$$T'(s) = \kappa(s) \left\{ a \cos u + b \sin u - \int_0^s \cos[u(s) - u(\delta)] T(t) \kappa(t) dt \right\},$$

where $u(s) = \int_0^s \tau(t) dt$. Finally, he note that the Frenet pattern is characteristic for a spherical curve in E^n and the method extends to E^n Frenet curves through successive application of transformations of the form $\int_0^s A(t) dt$. By using the same method, similar characterizations of time-like and space-like spherical curves lying on a Lorentzian sphere in E_1^4 have been given by the authors in [3].

On the other hand, the differential equation characterizing a hyperbolic spherical curve in Minkowski 3-space E_1^3 is given by

$$\frac{d}{ds} \left\{ \frac{1}{\tau(s)} \frac{d\rho}{ds} \right\} = \rho(s) \tau(s)$$

with $\rho(s) = 1/\kappa(s) \neq 0$ and $1/\tau(s) \neq 0$ [7]. An explicit solution of this differential equation is also given in [7], namely,

(i) if $\alpha(s)$ is a unit speed space-like curve with a space-like principal normal N , then

$$\rho(s) = a \sinh \left[\int_0^s \tau(t) dt \right] - b \cosh \left[\int_0^s \tau(t) dt \right],$$

(ii) if $\alpha(s)$ is a unit speed space-like curve with a time-like principal normal N , then

$$\rho(s) = -a \sinh \left[\int_0^s \tau(t) dt \right] + b \cosh \left[\int_0^s \tau(t) dt \right],$$

(iii) if $\alpha(s)$ is a unit speed space-like curve with a null principal normal N , then

$$a \sinh \left[\int_0^s \tau(t) dt \right] - b \cosh \left[\int_0^s \tau(t) dt \right] = 1,$$

where a and b are arbitrary constants.

We should note that there are no time-like and light-like (null) unit speed curves which lie on the hyperbolic sphere in Minkowski 3-space E_1^3 [7].

It is also shown in [7] that a unit speed space-like curve in E_1^3 with a space-like principal normal N (resp., with a time-like principal normal N) lies on a hyperbolic sphere if and only if there is a function $f \in C^1$ such that

$$f\tau = (1/\kappa)', \quad f' - \tau/\kappa = 0, \quad \text{and } 1/\kappa < |f| \quad (\text{resp., } 1/\kappa > |f|).$$

Putting $\rho = 1/\kappa$ we get

$$\rho' = \tau f, \quad f' = \tau \rho \quad \text{and} \quad \rho < |f| \quad (\text{resp.}, \quad \rho > |f|).$$

Thus, by considering the Frenet equations in the Lorentzian space, we obtain the following important results:

- (i) The characteristic of a hyperbolic spherical curve in E_1^4 is the Frenet Pattern. Thus, the problem of getting an integral characterization to a hyperbolic spherical curve is nothing else but integration of Frenet equations. In particular, Lorentzian correspondence of the Breuer-Gottlieb characterization is isomorphic to the integral characterization of Frenet equations in the Lorentzian plane.
- (ii) The methods used to obtain characterizations of hyperbolic spherical curves can be extended to include Frenet equations in E_1^3 .

2 Preliminaries

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

$$g_0 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4 . Since g_0 is an indefinite metric, an arbitrary vector $\mathbf{v} \in E_1^4$ can have one of three causal characters: it can be space-like if $g_0(\mathbf{v}, \mathbf{v}) > 0$ or $\mathbf{v} = \mathbf{0}$, time-like if $g_0(\mathbf{v}, \mathbf{v}) < 0$, and null (light-like) if $g_0(\mathbf{v}, \mathbf{v}) = 0$ and $\mathbf{v} \neq \mathbf{0}$. Similarly, an arbitrary curve $\alpha(s)$ in E_1^4 is locally space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are space-like, time-like or null, respectively. A space-like or time-like curve $\alpha(s)$ is said to be parameterized by arc length function s , if $g_0(\alpha'(s), \alpha'(s)) = 1$ or -1 [6]. The norm of a vector $\mathbf{v} \in E_1^4$ is given by $\|\mathbf{v}\| = \sqrt{|g_0(\mathbf{v}, \mathbf{v})|}$. Thus, $\mathbf{v}$ is a unit vector if $g_0(\mathbf{v}, \mathbf{v}) = \pm 1$. Two vectors $\mathbf{v}, \mathbf{w}$ in E_1^4 are said to be orthogonal if $g_0(\mathbf{v}, \mathbf{w}) = 0$. The velocity of a curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

The hyperbolic sphere with center $\mathbf{m} = (m_1, m_2, m_3, m_4) \in E_1^4$ and radius $r \in \mathbb{R}^+$ in the space-time E_1^4 is the hyper-quadric given by

$$H_0^3 = \left\{ \mathbf{a} = (a_1, a_2, a_3, a_4) \in E_1^4 \mid g_0(\mathbf{a} - \mathbf{m}, \mathbf{a} - \mathbf{m}) = -r^2 \right\}.$$

Let us denote the moving Frenet frame along the space-like curve $\alpha(s)$ in the space-time E_1^4 by $\{\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4\}$. Then $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ and $\mathbf{T}_4$ are the tangent, the principal normal, the first binormal and second binormal vector fields, respectively. In this Frenet frame, $\mathbf{T}_1, \mathbf{T}_2$ and $\mathbf{T}_3$ are space-like vectors, and $\mathbf{T}_4$ is time-like vector (the fourth orthonormal time-like vector to $\mathbf{T}_1 = \mathbf{T}$, $\mathbf{T}_2 = \mathbf{N}$, $\mathbf{T}_3 = \mathbf{B}$). Now, let $\alpha(s)$ be a space-like curve in the space-time E_1^4 , parameterized

by arc length function s . Then the Frenet formulae (T_4 time-like) for $\alpha(s)$ is given in [8]:

$$\begin{bmatrix} T_1' \\ T_2' \\ T_3' \\ T_4' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \mu \\ 0 & 0 & \mu & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}, \quad (2)$$

where $\{T_1, T_2, T_3, T_4\}$ are mutually orthogonal vectors satisfying equations

$$g_0(T_1, T_1) = g_0(T_2, T_2) = g_0(T_3, T_3) = 1, \quad g_0(T_4, T_4) = -1,$$

and $\kappa = \kappa(s)$, $\tau = \tau(s)$ and $\mu = \mu(s)$ are the first, the second, and the third curvature functions of curve $\alpha(s)$. $\mu(s) = g_0(B', T_4)$ measures the change of direction of the space spanned by $T_1 = T$, $T_2 = N$, $T_3 = B$.

3 H_0^3 Curves and E_1^3 Space-like Frenet Curves

Extension of Petrovic-Torgasev and Sucurovic's condition [7] to E_1^4 reveals the connection between H_0^3 space-like curves and E_1^3 space-like Frenet curves (a similar extension given by Dannon in [2]). Then we have:

Proposition 1. Let $\alpha(s)$ be a E_1^4 unit speed C^5 space-like Frenet curve with curvature functions $\kappa(s)$, $\tau(s)$, $\mu(s)$. Then the followings are equivalent:

(i) $\alpha(s)$ lies on a hyperbolic sphere H_0^3 in E_1^4 .

(ii) $\kappa(s) \neq 0$ and there are two C^2 functions $f(s)$ and $g(s)$ such that

$$\rho' = \tau f, \quad f' = -\tau\rho + \mu g \quad (\rho = 1/\kappa), \quad g' = \mu f, \quad \text{and} \quad \rho^2 + f^2 < g^2. \quad (3)$$

Proof. (i) $\Rightarrow$ (ii): Assume that $\alpha(s)$ lies on a E_1^4 hyperbolic sphere of radius r and center x_0 (assuming at the origin). By assumption we have $g_0(\alpha, \alpha) = -r^2$. Differentiations with respect to arc length s of $g_0(\alpha, \alpha) = -r^2$ give first $g_0(\alpha, T_1) = 0$, and then $g_0(\alpha, T_1') + g_0(T_1, T_1) = 0$, i.e. $g_0(\alpha, T_1') + 1 = 0$. This shows that $g_0(\alpha, T_1')$ is nowhere zero. Then using Frenet equations give $\kappa(s) \neq 0$ and $-g_0(\alpha, T_2) = 1/\kappa = \rho$. By a new differentiation we find $\rho' = -\tau g_0(\alpha, T_3)$. Let us define $f = -g_0(\alpha, T_3)$, then we have $\rho' = \tau f$. Differentiation of $f = -g_0(\alpha, T_3)$ and using Frenet equations, we get $f' = -\tau\rho + \mu g$, where $g = -g_0(\alpha, T_4)$. Thus $f' = -\tau\rho + \mu g$. Finally, differentiation of $g = -g_0(\alpha, T_4)$ and using Frenet equations, we obtain $g' = \mu f$. Thus we obtain

$$\rho' = \tau f, \quad f' = -\tau\rho + \mu g \quad (\rho = 1/\kappa), \quad g' = \mu f.$$

We can write these equations as

$$\begin{bmatrix} \rho' \\ f' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & \tau & 0 \\ -\tau & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix} \begin{bmatrix} \rho \\ f \\ g \end{bmatrix}.$$

Now we can write

$$\alpha = g_0(\alpha, T_1)T_1 - \rho T_2 - f T_3 + g T_4 = -\rho T_2 - f T_3 + g T_4.$$

Then the equation $g_0(\alpha, \alpha) = -r^2$ gives the inequality $\rho^2 + f^2 - g^2 < 0$.

(ii) $\Rightarrow$ (i): Assume that $\alpha(s)$ satisfies the condition (ii). Let us consider the curve $\gamma(s)$ defined by

$$\gamma(s) = (\alpha + \rho T_2 + f T_3 - g T_4)(s).$$

Then using the equations (2) and (3) gives $\gamma'(s) = 0$. Thus $\gamma(s) = \text{const.} \equiv x_0$.

Thus we have $\alpha - x_0 = -\rho T_2 - f T_3 + g T_4$. Using the orthogonality gives

$$g_0(\alpha - x_0, \alpha - x_0) = \rho^2 + f^2 - g^2.$$

Differentiation of this gives $g_0'(\alpha - x_0, \alpha - x_0) = 0$, so $\rho^2 + f^2 - g^2 = \text{const.} = -r^2$, since $\rho^2 + f^2 - g^2 < 0$. Therefore $g_0(\alpha - x_0, \alpha - x_0) = -r^2$, that is, $\alpha(s)$ lies on the hyperbolic sphere of radius r with center x_0 . ■

Note that, the equations (3) are the Frenet formulae of a E_1^3 -space-like curve with time-like $T_3 = B$. Thus E_1^4 hyperbolic spherical curve equations have the structure of E_1^3 space-like Frenet curves. Therefore, finding an integral characterization for a E_1^4 hyperbolic spherical curve is identical to finding it for a E_1^3 space-like Frenet curve.

4 An Integral Characterizations of E_1^3 Frenet Curves

Extending the methods used to obtain characterizations of hyperbolic spherical curves, we obtain the integral characterization of an E_1^3 space-like Frenet curve as follows. The two bottom E_1^3 space-like Frenet equations are of the form

$$g'(s) = -\lambda f + \mu h, \quad h'(s) = \mu g.$$

Assuming that μ is non vanishing (the conclusions are free from this assumption), then we get

$$g'/\mu = -(\lambda f/\mu) + h.$$

Differentiation of this, then substitution of h' and eventually application of the change of variable $u(s) = \int_0^s \mu(t) dt$, reduce this equation to

$$g'' - g = -(\lambda f/\mu)',$$

where the variable is u . Then we can easily find a particular solution for $g(u)$ as

$$-\int_0^s \cosh[u(s)-u(t)] \lambda(t) f(t) dt.$$

Then we have the following Theorem:

Theorem 1. Let $\alpha(s)$ be a C^4 space-like curve in E_1^3 , parameterized by its arc length s with $\alpha'(s) = T$. Then the following are equivalent:

(i) $\alpha(s)$ has a Frenet system T, N, B , curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy the Frenet equations

$$T'(s) = \kappa N, \quad N'(s) = -\kappa T + \tau B, \quad B'(s) = \tau N.$$

(ii) There are constant vectors a and b such that

$$T'(s) = \kappa(s) \left\{ a e^u + b e^{-u} - \int_0^s \cosh[u(s)-u(t)] T(t) \kappa(t) dt \right\},$$

where $u(s) = \int_0^s \tau(t) dt$.

Proof. (ii) $\Rightarrow$ (i): Assume that the condition holds. Let us define the functions $N(s)$ and $B(s)$ as follows:

$$N(s) = -\int_0^s \cosh[u(s)-u(t)] T(t) \kappa(t) dt + a e^u + b e^{-u}$$

and

$$B(s) = -\int_0^s \sinh[u(t)-u(s)] T(t) \kappa(t) dt + a e^u - b e^{-u}.$$

Then it is easy to check that T, N and B satisfy the Frenet equations.

(i) $\Rightarrow$ (ii): Assume that the equations hold. Then the above $N(s)$ and $B(s)$ solve the coupled system

$$N' = -\kappa T + \tau B, \quad B' = \tau N.$$

The last equation $\kappa N = T'$ is our condition. ■

This method may be extended to E_1^n space-like Frenet curves through successive application of transformations of the form $\int_0^s A(t) dt$ by using the terminology of [4].

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