Pure Mathematical Sciences, Vol. 3, 2014, no. 2, 97 - 103
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http://dx.doi.org/10.12988/pms.2014.438

Random Measures Do Produce Generalized Likelihoods

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Abstract

In this article, we randomize the notion of the likelihood function by using random variables which may come from random measures. This randomization may be applied all over the positions of a sample and it is related to the notion of the base of a cone.

Mathematics Subject Classification: 60G57; 97K50

Keywords: random measure; base of a cone; generalized likelihood function; randomization

1 Random measures and some comments on their properties

We remind the definition given in [3] (this is the Deinition [3, Def.1.1] slightly altered) of the random measure. We consider a probability space $(\Omega, \mathcal{F}, \mu)$ and a measurable space $(E, \mathcal{E})$.

Definition 1.1 *A random measure ξ on $(E, \mathcal{E})$ over the probability space $(\Omega, \mathcal{F}, \mu)$ is a map*

$$\xi : \mathcal{E} \times \Omega \to \boldsymbol{R}_+,$$

such that

1. *the map* $\omega \mapsto \xi(A, \omega)$ *is a random variable for any* $A \in \mathcal{E}$

2. *the map* $A \mapsto \xi(A, \omega)$ *is a (probability) measure on* $\mathcal{E}$, μ*- almost surely in* Ω.

What kind of random variables are the $\xi_A : \Omega \to \mathbf{R}_+, A \in \mathcal{E}$ which are defined above? First of all, we may suppose that they belong to the linear space of the measurable $\mathbf{R}$-valued random variables $L^0(\Omega, \mathcal{F}, \mu)$ with respect to the probability space $(\Omega, \mathcal{F}, \mu)$. We also may suppose that these random variables $\xi_A, A \in \mathcal{E}$ also belong to $L^1(\Omega, \mathcal{F}, \mu)$. Since by definition the realizations-values of these variables are positive real numbers, they actually belong to $L^1_+(\Omega, \mathcal{F}, \mu)$. Without loss of generality we may suppose that ξ_ω is a probability measure on $\mathcal{E}$ for any $\omega \in \Omega$. If we suppose that E is a compact metric space, we may suppose that $\mathcal{E}$ is the Borel σ-algebra generated by it and we may consider the space of the σ-additive measures $ca(\mathcal{E})$ over the σ-algebra $\mathcal{E}$. This space is endowed with the total variation norm, according to which the set of the probability measures on $\mathbf{E}$ is the following set $\{\nu \in ca_+(\mathcal{E}) | \|\nu\| = 1\}$, namely $\{\nu \in ca_+(\mathcal{E}) | |\nu|(E) = \nu(E) = 1\}$. This set is the base of the cone $ca_+(\mathcal{E})$ defined by $\mathbf{1}$. But the space $ca(\mathcal{E})$ is the dual space of the space $C(E)$ of the continuous real-valued functions on E. This implies the definition of a random variable $\omega \mapsto \langle f, \xi_\omega \rangle, f \in C(E)$, which is denoted by $X_{f,\xi}$. Namely, $X_{f,\xi}(\omega) = \langle f, \xi_\omega \rangle = \int_E f(t) d\xi_\omega(t), \omega \in \Omega$. We also may supose that $(E, \mathcal{E}) = (\Omega, \mathcal{F})$, while Ω is a compact metric space and $\mathcal{F}$ its Borel σ-algebra. In this case, ξ_ω is an element of the base $\{\nu \in ca_+(\mathcal{F}) | |\nu|(\Omega) = \nu(\Omega) = 1\}$. Hence by definition, $\int_\Omega d\xi_\omega(t) = 1, \omega \in \Omega$. Also, we may define the random variable $X_{f,\xi}$ which may be taken to belong either to $L^0(\Omega, \mathcal{F}, \mu)$ or to $L^1(\Omega, \mathcal{F}, \mu)$ and specifically to $L^1(\Omega, \mathcal{F}, \mu)$ for any $f \in C(\Omega)$.

Lemma 1.2 *If* $(E, \mathcal{E}) = (\Omega, \mathcal{F})$*, then if* $\xi_A, A \in \Omega$ *and* $X_{f,\xi}, \forall f \in C(\Omega)$ *belong to* $L^1(\Omega, \mathcal{F}, \mu)$*, we may define measures on* $(\Omega, \mathcal{F})$ *which are absolutely continuous with respect to* μ.

Proof: If we take a certain $A \in \mathcal{F}$ and we consider the map $Q_{\xi_A}(B) = \int_B \xi_A(\omega) d\mu(\omega), B \in \mathcal{F}$. In the same way, we may take a certain $f \in C(\Omega)$ and we may also consider the map $Q_{X_{f,\xi}}(B) = \int_B X_{f,\xi}(\omega) d\mu(\omega), B \in \mathcal{F}$. As it is well-known the above maps define measures of bounded variation, or else $Q_{\xi_A}, Q_{X_{f,\xi}} \in ca(\mathcal{F})$, which are absolutely continuous with respect to μ. This implies the existence of the Radon -Nikodym derivatives $\frac{dQ_{\xi_A}}{d\mu}, \frac{dQ_{X_{f,\xi}}}{d\mu}$, see [1, Th.12.18].

Finally, we may also suppose that the measures $Q_{\xi_A}, Q_{X_{f,\xi}} \in ca(\mathcal{F})$ are actually probability measures, but this is not always necessary.

2 Generalized likelihood functions

As it is well-known from elementary statistics, a random sample of length n is a vector $(X_1, X_2, ..., X_n)$ of independent, identically distributed random variables $X_i, i = 1, 2, ..., n$ whose common cumulative distribution function is F. In terms of the probability space $(\Omega, \mathcal{F}, \mu)$, the distribution probability measure $Q_{X_i}, i = 1, 2, ..., n$ of the random variable $X_i \in L^1(\Omega, \mathcal{F}, \mu)$ is defined on $(\Omega, \mathcal{F})$ is defined as follows: $Q_{X_i}(B) = \mu(X_i^{-1}(B)), B \in \mathcal{B}_{\mathbf{R}}, i = 1, 2, ..., n$. Since the $X_i, i = 1, 2, ..., n$ formulate a random sample, we have that $Q_{X_i} = Q$ for some $Q \in \{\nu \in ca_+(\mathcal{B}_{\mathbf{R}}) | \nu(\mathbf{R}) = 1\}$. From the independence of the random variables $X_i, i = 1, 2, ..., n$ we may define the product distribution measure $Q_{X_1} \otimes Q_{X_2} ... \otimes Q_{X_n}(B_1 \times B_2 \times ... \times B_n) = Q_{X_1}(B_1) Q_{X_2}(B_2) ... Q_{X_n}(B_n)$, where $B_i \in \mathcal{B}_{\mathbf{R}}$, because $\mathcal{B}_{\mathbf{R}} \times \mathcal{B}_{\mathbf{R}} ... \times \mathcal{B}_{\mathbf{R}}$ is a class of subsets of the Borel σ-algebra of $\mathbf{R}^n$ which contains the open subsets of $\mathbf{R}^n$.

Hence, we introduce the notion of (I, ξ_I)-*randomization*, where

$$I \subseteq \{1, 2, ..., n\}$$

is a subset of indices $\{i_1, i_2, ..., i_k\}$ with $i_1 < i_2 < ... < i_k, 1 \leq k \leq n$, and $\xi_I = (\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k})$, where each $\xi_{i_j}, j = 1, 2, ..., k$ is a random measure on $(\Omega, \mathcal{F})$ over $(\Omega, \mathcal{F}, \mu)$. Specifically, (I, ξ_I)-randomization obtains two expressions. The first expression is the (I, ξ_I, A_I)-*randomization* and the second expression is the (I, ξ_I, f_I)-*randomization*.

Remark 2.1 *By the first expression we mean that the $Q_{X_{i_j}}$ distribution measure is replaced by the probability measure $Q_{\xi^{i_j}_{A_{i_j}}}, A_{i_j} \in \mathcal{F}, j = 1, 2, ..., k$, beause they are approximated by them. By the second expression we mean that the $Q_{X_{i_j}}$ distribution measure is replaced by the probability measure $Q_{X_{\xi^{i_j}, f_{i_j}}}, f_{i_j} \in C(\Omega), j = 1, 2, ..., k$, for the same reason.*

Remark 2.2 *The results concerning the (I, ξ_I, A_I) or the (I, ξ_I, f_I) randomization cope with the approximation either of the $Q_{X_{i_j}}$ via $Q_{\xi^{i_j}_{A_{i_j}}}, A_{i_j} \in \mathcal{F}$ or by $Q_{X_{\xi^{i_j}, f_{i_j}}}, f_{i_j} \in C(\Omega)$, $j = 1, 2, ..., k$, respectively.*

Theorem 2.3 *Consider a locally compact topological space Ω and let $\mathcal{F}$ be its Borel σ-algebra. Then for any $X \in L^1(\Omega, \mathcal{F}, \mu)$, where μ is a probability measure on $(\Omega, \mathcal{F})$, there is at least one random measure $\xi : \mathcal{F} \times \Omega \to \overline{\boldsymbol{R}}$, a sequence $(f_n)_{n \in \boldsymbol{N}} \subseteq L^1$ depending on X and a sequence of sets $(A_n)_{n \in \boldsymbol{N}} \subseteq \mathcal{F}$, such that $\frac{dQ_{(\xi, A_n)}}{d\mu} \xrightarrow{L^1} \frac{dQ_X}{d\mu}$ and $\frac{dQ_{X_{(f_n, \xi)}}}{d\mu} \xrightarrow{L^1} \frac{dQ_X}{d\mu}$.*

Proof: It is well-known that $C_c(\Omega)$ (being the space of the continuous functions with compact support) is $\|.\|_1$-dense in $L^1(\Omega,\mathcal{F},\mu)$, see [1, Th.12.9]. Thus, for any n, we deduce the existence of some $f_n \in C_c(\Omega)$ such that

$$\|f_n - \frac{dQ_X}{d\mu}\|_1 < \frac{1}{n}.$$

Hence, we have to find $\tilde{f}_n \in L^1$ and a random measure ξ such that $f_n(\omega) = \left\langle \tilde{f}_n, \xi_\omega \right\rangle$, μ-a.e. If we consider some ξ for which $\xi_\omega = \mu$, μ-a.e., then we may put $\tilde{f}_n = f_n$. Also, if we suppose that $\xi(A) \in L^1, A \in \mathcal{F}$, then for the second case we select $A_n, n \in \mathbf{N}$ to be the sets $A_n = \xi^{-1}(f_n) \in \mathcal{F}$.

Corollary 2.4 *Consider a locally compact topological space Ω and let $\mathcal{F}$ be its Borel σ-algebra. Then for any $X \in L^2(\Omega,\mathcal{F},\mu)$, where μ is a probability measure on $(\Omega,\mathcal{F})$, there is at least one random measure $\xi : \mathcal{F}\times\Omega \to \overline{\boldsymbol{R}}$, a sequence $(f_n)_{n\in\boldsymbol{N}} \subseteq L^2$ depending on X and a sequence of sets $(A_n)_{n\in\boldsymbol{N}} \subseteq \mathcal{F}$, such that $\frac{dQ_{(\xi,A_n)}}{d\mu} \xrightarrow{L^2} \frac{dQ_X}{d\mu}$ and $\frac{dQ_{X_{(f_n,\xi)}}}{d\mu} \xrightarrow{L^1} \frac{dQ_X}{d\mu}$.*

Proof: [1, Th.12.9] also holds for $L^2(\Omega,\mathcal{F},\mu)$.

According to Remark 2.1, we may re-state more exact a definition of (I,ξ_I,A_I)-*randomization* and (I,ξ_I,f_I)-*randomization*.

Definition 2.5 *We obtain a (I,ξ_I,A_I)-randomization with respect to the set I and for the sample $(X_1,X_2,...,X_n)$, if we replace the density $\frac{dQ_{X_{i_j}}}{d\mu}$ by $\frac{dQ_{X_{\xi^{i_j}_{A_{i_j,n}}}}}{d\mu}$ for any $j=1,2,...,k$, $n\in\boldsymbol{N}$.*

Definition 2.6 *We obtain a (I,ξ_I,f_I)-randomization with respect to the set I and for the sample $(X_1,X_2,...,X_n)$, if we replace the density $\frac{dQ_{X_{i_j}}}{d\mu}$ by $\frac{dQ_{X_{\xi^{i_j},f_{i_j,n}}}}{d\mu}$ for any $j=1,2,...,k$, $n\in\boldsymbol{N}$.*

Definition 2.7 *The (I,ξ_I,f_I)-**generalized likehihood** with respect to the sample*

$$(X_1,X_2,...,X_n)$$

and the index set $I=\{1,2,...,k\}$ which concerns the positions of randomization, is the sequence of joint densities

$$\xi(I,f) = \prod_{j=1}^{k} \frac{dQ_{X_{\xi^{i_j},f_{i_j,n}}}}{d\mu}, n\in\boldsymbol{N}.$$

Definition 2.8 *The* (I, ξ_I, A_I)*-**generalized likehihood** with respect to the sample*

$$(X_1, X_2, ..., X_n)$$

and the index set $I = \{1, 2, ..., k\}$ *which concerns the positions of randomization, is the sequence of joint densities*

$$\xi(I, A) = \prod_{j=1}^{k} \frac{dQ_{X_{\xi^{i_j}_{A_{i_j,n}}}}}{d\mu}, n \in \boldsymbol{N}.$$

In the next Propositions, we suppose that Ω is locally compact topological space.

Proposition 2.9

$$\xi(I, f) \xrightarrow{L^1} \prod_{j=1}^{k} \frac{dQ_{X_{i_j}}}{d\mu}, n \to +\infty,$$

if the k *factors of the product belong to* L^∞*.*

Proof: It arises from the Corollary 2.4 and the Hölder Inequality.

Proposition 2.10

$$\xi(I, A) \xrightarrow{L^1} \prod_{j=1}^{k} \frac{dQ_{X_{i_j}}}{d\mu}, n \to +\infty,$$

if the k *factors of the product belong to* L^∞*.*

Proof: It arises from the Corollary 2.4 and the Hölder Inequality.

3 Appendix

In this Section, we give some essential notions and results from the theory of partially ordered linear spaces which are used in this paper. For these notions and definitions, see [2, Ch.1, Ch.2, Ch.3]. Let E be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbf{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(E, \geq)$ where E is a linear space and $\geq$ is a binary relation on E satisfying the following properties:

(i) $x \geq x$ for any $x \in E$ (reflexive)

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive) *(iii)* If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbf{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of E),

is called *partially ordered linear space*. The binary relation $\geq$ in this case is a *partial ordering* on E. The set $P = \{x \in E | x \geq 0\}$ is called *(positive) wedge* of the partial ordering $\geq$ of E. Given a wedge C in E, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \Longleftrightarrow x - y \in C,$$

is a partial ordering on E, called *partial ordering induced by C on E*. If the partial ordering $\geq$ of the space E is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then P is a cone.

E' denotes the linear space of all linear functionals of E, called *algebraic dual* while E^* is the norm dual of E, in case where E is a normed linear space.

Suppose that C is a wedge of E. A functional $f \in E'$ is called *positive functional* of C if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a *strictly positive functional* of C if $f(x) > 0$ for any $x \in C \setminus \{0\}$. A linear functional $f \in E'$ where E is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any $x \in C$. In case where a uniformly monotonic functional of C exists, C is a cone. $C^0 = \{f \in E^* | f(x) \geq 0 \text{ for any } x \in C\}$ is the *dual wedge of C in E^**. Also, by C^{00} we denote the subset $(C^0)^0$ of E^{**}. It can be easily proved that if C is a closed wedge of a reflexive space, then $C^{00} = C$. If C is a wedge of E^*, then the set $C_0 = \{x \in E | \hat{x}(f) \geq 0 \text{ for any } f \in C\}$ is the *dual wedge of C in E*, where $\hat{}: E \to E^{**}$ denotes the natural embedding map from E to the second dual space E^{**} of E. Note that if for two wedges K, C of E, $K \subseteq C$ holds, then $C^0 \subseteq K^0$.

If C is a cone, then a set $B \subseteq C$ is called *base* of C if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_x x \in B$. The set $B_f = \{x \in C | f(x) = 1\}$ where f is a strictly positive functional of C is the *base of C defined by f*. B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0 \notin \overline{B}$ then C is called *well-based*. If C is well-based, then a bounded base of C defined by a $g \in E^*$ exists. If $E = C - C$ then the wedge C is called *generating*, while if $E = \overline{C - C}$ it is called *almost generating*. If C is generating, then C^0 is a cone of E^* in case where E is a normed linear space. Also, $f \in E^*$ is a uniformly monotonic functional of C if and only if $f \in intC^0$, where $intC^0$ denotes the norm-interior of C^0.

References

[1] C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis, A Hitchhiker's Guide,* (second edition), Springer, 1999.

[2] G. Jameson, *Ordered Linear Spaces, Lecture Notes in Mathematics, Vol.141,* Springer-Verlag, 1970.

[3] A. Karr, Lévy Random Measures, *The Annals of Probability*, **6** (1978), 57-71.

Received: March 18, 2014