Superderivation of the Hyperbolic Subalgebra for $\text{pl}(m, n, R)$

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Abstract

Let R be an arbitrary commutative ring with 1, $1/2 \in R$, $\text{pl}(m, n, R)$ the general linear Lie superalgebra over R . The aim of this paper is to give an explicit description of the superderivation algebras of the hyperbolic subalgebra of $\text{pl}(m, n, R)$.

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1 Introduction

If $\deg(x)$ occurs in some expression, then x is assumed to be a homogeneous element denoted by $\text{hg}(x)$ and $\deg(x)$ is the degree of x about Z_2 -gradation.

Let R be a commutative ring with 1, $1/2 \in R$ and L a Lie superalgebra over R . Recall that an R -linear mapping $\phi : L \rightarrow L$ is called a Z_2 -homogeneous Lie superderivation of degree θ if

$$\phi([a, b]) = [\phi(a), b] + (-1)^{\deg(a)\theta} [a, \phi(b)] \quad \forall a, b \in L_{\bar{0}} \cup L_{\bar{1}}.$$

A Lie superderivation is the sum of a Lie superderivation of degree $\bar{0}$ and a Lie superderivation of degree $\bar{1}$.

In recently years, many significant researches have been done in derivations of general linear Lie algebra and its subalgebras (see [1-2]). In this paper, we will describe the hyperbolic subalgebras of $\text{pl}(m, n, R)$ and give an explicit description on the superderivations for the hyperbolic subalgebras of $\text{pl}(m, n, R)$ to $\text{pl}(m, n, R)$. We obtain that any Lie superderivation ϕ for the hyperbolic subalgebras of $\text{pl}(m, n, R)$ to $\text{pl}(m, n, R)$ can be uniquely written as the sum of an inner superderivation and an induced superderivation.

2 The hyperbolic subalgebras of $\text{pl}(m, n, R)$

Put $l = m + n$. Throughout this paper, $\text{pl}(m, n, R)$ denotes the R -algebra of all $l \times l$ matrices over a commutative ring R . It is obvious that $\{E_{ij} \mid 1 \leq i, j \leq l\}$ is a basis of $\text{pl}(m, n, R)$, where E_{ij} is the matrix whose sole nonzero entry is 1 in the (i, j) position. The bracket operation

$$[E_{ij}, E_{kt}] = \delta_{jk} E_{it} - (-1)^{(\tau(i)+\tau(j))(\tau(k)+\tau(t))} \delta_{it} E_{kj},$$

where $\tau(i) = \bar{0}$ for $i \in \{1, 2, \dots, m\}$, $\tau(i) = \bar{1}$ for $i \in \{m+1, m+2, \dots, l\}$.

Define $\deg(E_{ij}) = \tau(i) + \tau(j)$ on $\text{pl}(m, n, R)$ a structure of Lie superalgebra over R . We denote by $D(m, n, R)$ the Lie superalgebra consisting of all diagonal matrices.

Definition 2.1 Let $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i, j \leq l\}$ be a subset of $I(R)$ consisting of l^2 ideals of R . We call Φ a flag of ideals of R if and only if for any i, j, k :

- (1) $A_{i,i} = R, i = 1, 2, \dots, l$;
- (2) $A_{i,k} A_{k,j} \subseteq A_{i,j}$.

Example 2.2 If $i \neq j$, let $A_{i,i} = R$ for $i = 1, 2, \dots, l$. Then $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i, j \leq l\}$ is a flag of ideals of R .

Example 2.3 If all $A_{i,j}, 1 \leq i, j \leq l$ are taken to be R , then $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i, j \leq l\}$ is a flag of ideals of R .

Example 2.4 Set $A_{i,i} = R$ for $i = 1, 2, \dots, l$. Let $A_{1,2}, A_{2,3}, \dots, A_{l-1,l}, A_{2,1}, A_{3,2}, \dots, A_{l,l-1}$ be any ideals of R respectively, and set

$$A_{i,j} = \prod_{k=1}^{j-i} A_{i+k-1} A_{i+k}, \quad 1 \leq i < j \leq l,$$

$$A_{i,j} = \prod_{k=1}^{j-i} A_{i-k+1} A_{i-k}, \quad 1 \leq j < i \leq l.$$

Then $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i, j \leq l\}$ is a flag of ideals of R .

Definition 2.5 Any subalgebra of $pl(m, n, R)$ containing $D(m, n, R)$ is called a hyperbolic subalgebra.

Lemma 2.6 L is a hyperbolic subalgebra of $pl(m, n, R)$ if and only if there exists a flag $\Phi = \{A_{i,j} \in I(R) | 1 \leq i, j \leq l\}$ of ideals of R such that $L = L_\Phi = \sum_{1 \leq i, j \leq l} A_{i,j} E_{i,j}$.

3 Main results

Let $L_\Phi = \sum_{1 \leq i, j \leq l} A_{i,j} E_{i,j}$ be a fixed hyperbolic subalgebra of $pl(m, n, R)$ with $\Phi = \{A_{i,j} \in I(R) | 1 \leq i, j \leq l\}$ a flag of ideals of R . In order to show any superderivation of L_Φ to $pl(m, n, R)$, in this section we construct certain superderivations of L_Φ to $pl(m, n, R)$ as follows.

(I) Inner superderivation

Let $x \in pl(m, n, R)$, then $\text{ad} : L_\Phi \rightarrow pl(m, n, R), y \mapsto [x, y]$ is a superderivation of L_Φ to $pl(m, n, R)$, called the inner superderivation of L_Φ to $pl(m, n, R)$ induced by x .

(II) Induced superderivation

Let $\Sigma = \{\sigma_{i,j} \in \text{Hom}_R(A_{i,j}, R) | 1 \leq i, j \leq l\}$ be a set consisting of l^2 homomorphisms of R -modules. We call Σ suitable for induced superderivations if for any i, j, k :

- (1) $\sigma_{i,i}(r) = \lambda_i r$, where $\lambda_i, r \in R$,
- (2) $\sigma_{i,j}(a_{i,k} a_{k,j}) = \sigma_{i,k}(a_{i,k}) a_{k,j} + a_{i,k} \sigma_{k,j}(a_{k,j})$. where $a_{i,k} \in A_{i,k}, a_{k,j} \in A_{k,j}$.

Using the homomorphism $\Sigma = \{\sigma_{i,j} \in \text{Hom}_R(A_{i,j}, R) | 1 \leq i, j \leq l\}$ which is suitable for induced superderivations, we define $\Phi_\Sigma : L_\Phi \rightarrow pl(m, n, R)$ by sending any $\sum_{1 \leq i, j \leq l} a_{i,j} E_{i,j}$ to $\sum_{1 \leq i, j \leq l} \sigma_{i,j}(a_{i,j}) E_{i,j}$.

Lemma 3.1 Φ_Σ is a Lie superderivation of L_Φ to $pl(m, n, R)$.

If $l = 1$, there is nothing to do, so we only consider the case $l \geq 2$. Let W be the subset of $pl(m, n, R)$ consisting of all the matrices whose diagonal entries are all 0. In the following, we describe any Lie superderivation of the hyperbolic subalgebra L_Φ to $pl(m, n, R)$.

Theorem 3.2 Let R be a commutative ring with $1, 1/2 \in R$, $L_\Phi = \sum_{1 \leq i, j \leq l} A_{i,j} E_{i,j}$ a fixed hyperbolic subalgebra of $pl(m, n, R)$ containing $D(m, n, R)$ with $\Phi = \{A_{i,j} \in I(R) | 1 \leq i, j \leq l\}$ a flag of ideals of R . Then each superderivation of L_Φ to $pl(m, n, R)$ may be uniquely written as the sum of an induced superderivation and an inner superderivation induced by an element in W .

References

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