Some Fixed Point Theorems in Dislocated Metric Spaces

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Abstract

In this paper, we prove a contractive type condition with rational expression and generalize common fixed point theorems for Expansive Type Mapping in Dislocated metric space. Our results extends and generalizes many well known results.

Keywords: Dislocated metric space; fixed point; dq-Cauchy sequence.

1 Introduction

P. Hitzler et al., introduced the notation of dislocated metric spaces in which self distance of a point need not be equal to zero. They also generalized the famous Banach contraction principle in this space, and satisfying certain contractive conditions has been at the center of vigorous research activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. Aage et. al., established some

important fixed point theorems in single and pair of mappings in dislocated metric space. The purpose of this paper is to establish a common fixed point theorem for Expansive Type Mapping in complete dislocated metric space. Our result generalizes some results of fixed points. A property which extends and generalizes the well known Banach contraction principle and known results.

2 Preliminaries

Definition 2.1 : Let X be a nonempty set, let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying following conditions.

- (i) $d(x, y) = d(y, x) = 0$ implies $x = y$.
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
- (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

Then d is called a dislocated metric spaces or d -metric on X .

Definition 2.2 : A sequence $\{x_n\}$ in d -metric space (X, d) is said to be a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, implies $d(x_m, x_n) < \epsilon$.

Definition 2.3 : A sequence $\{x_n\}$ in d -metric space (X, d) is said to be a convergent to x if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

Definition 2.4 : A d -metric space (X, d) is said to be a Complete if every Cauchy sequence is convergent in X .

3 Main Results

Theorem 3.1 : Let (X, d) be a complete dislocated metric space. Let T be a continuous mapping from X to X satisfying the following condition:

$$d(Tx, Ty) + \alpha \left(\frac{d(y, Tx) + d(x, Ty)}{1 + d(y, Ty) + d(x, Ty)} \right) \geq \beta \left(\frac{d(y, Tx) + d(x, Ty)}{d(x, y) + d(y, Ty)} \right) d(y, Ty) + \gamma d(x, y) \quad \dots(1)$$

for all $x, y \in X, x \neq y$, where $\alpha, \beta, \gamma > 0$ are all real constants and $\beta + \gamma > 1 + 2\alpha$, $\gamma > 1 + 2\alpha$. Then T has a unique fixed point.

Proof: Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}, n \in \mathbb{N}$ as follows and $Tx_n = x_{n-1}$ for $n=1, 2, 3, \dots$. Then, using (1) we obtain

$$\begin{aligned}
d(Tx_{n+1}, Tx_{n+2}) + \alpha \left(\frac{d(x_{n+2}, Tx_{n+1}) + d(x_{n+1}, Tx_{n+2})}{1 + d(x_{n+2}, Tx_{n+2}) d(x_{n+1}, Tx_{n+2})} \right) \\
\geq \beta \left(\frac{d(x_{n+2}, Tx_{n+1}) + d(x_{n+1}, Tx_{n+2})}{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, Tx_{n+2})} \right) d(x_{n+2}, Tx_{n+2}) + \gamma d(x_{n+1}, x_{n+2}) \dots (1) \\
d(x_n, x_{n+1}) + \alpha \left(\frac{d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})}{1 + d(x_{n+2}, x_{n+1}) d(x_{n+1}, x_{n+1})} \right) \\
\geq \beta \left(\frac{d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})}{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+1})} \right) d(x_{n+2}, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})
\end{aligned}$$

$$d(x_n, x_{n+1}) + \alpha d(x_{n+2}, x_n) \geq \left(\frac{\beta}{2} \right) d(x_{n+2}, x_n) + \gamma d(x_{n+1}, x_{n+2})$$

$$\left(1 + \alpha - \frac{\beta}{2} \right) d(x_n, x_{n+1}) \geq \left(-\alpha + \frac{\beta}{2} + \gamma \right) d(x_{n+1}, x_{n+2})$$

$$d(x_{n+1}, x_{n+2}) \leq \left(\frac{1 + \alpha - \frac{\beta}{2}}{-\alpha + \frac{\beta}{2} + \gamma} \right) d(x_n, x_{n+1})$$

$$d(x_{n+1}, x_{n+2}) \leq h d(x_n, x_{n+1}) \dots (2) \text{ where } h = \left(\frac{1 + \alpha - \frac{\beta}{2}}{-\alpha + \frac{\beta}{2} + \gamma} \right) < 1$$

In the same way, we have $d(x_{n+1}, x_{n+2}) \leq h d(x_n, x_{n+1})$.

By (2), we get $d(x_{n+1}, x_{n+2}) \leq h^2 d(x_{n-1}, x_n)$. Continue this process, we get in general $d(x_n, x_{n+1}) \leq h^{n+1} d(x_1, x_0)$. Since $0 \leq h < 1$ as $n \rightarrow \infty$, $h^{n+1} \rightarrow 0$. Hence $\{x_n\}$ is a d-cauchy sequence in X . Thus $\{x_n\}$ dislocated converges to some u in X . Since T is continuous we have $T(u) = \lim T(x_n) = \lim x_{n+1} = u$. Thus u is a fixed point T .

Uniqueness: Let y^* be another fixed point of T in X , then $Ty^* = y^*$ and $Tx^* = x^*$.

$$\begin{aligned}
d(Tx^*, Ty^*) + \alpha \left(\frac{d(y^*, Tx^*) + d(x^*, Ty^*)}{1 + d(y^*, Ty^*) d(x^*, Ty^*)} \right) \\
\geq \beta \left(\frac{d(y^*, Tx^*) + d(x^*, Ty^*)}{d(x^*, y^*) + d(y^*, Ty^*)} \right) d(y^*, Ty^*) + \gamma d(x^*, y^*) \quad \dots(3) \\
d(x^*, y^*) + \alpha \left(\frac{d(y^*, x^*) + d(x^*, y^*)}{1 + d(y^*, y^*) d(x^*, y^*)} \right) \geq \beta \left(\frac{d(y^*, x^*) + d(x^*, y^*)}{d(x^*, y^*) + d(y^*, y^*)} \right) d(y^*, y^*) + \gamma d(x^*, y^*) \\
d(x^*, y^*) + 2\alpha d(x^*, y^*) \geq \gamma d(x^*, y^*) \\
d(x^*, y^*) \leq \left(\frac{1 + 2\alpha}{\gamma} \right) d(x^*, y^*)
\end{aligned}$$

This is true only when $d(x^*, y^*) = 0$. Similarly $d(y^*, x^*) = 0$. Hence $d(x^*, y^*) = d(y^*, x^*) = 0$ and so $x^* = y^*$. Hence T has a unique fixed point.

Theorem 3.2 : Let (X, d) be a complete dislocated metric space. Let T be a continuous mapping from X to X satisfying the following condition:

$$d(Tx, Ty) + \alpha \left(\frac{d(y, Tx) + d(x, Ty)}{1 + d(y, Ty) d(x, Ty)} \right) \geq \beta \left(\frac{d(x, Tx) d(y, Ty)}{d(x, y)} \right) + \gamma d(x, y) \dots\dots\dots(1)$$

for all

$x, y \in X, x \neq y$, where $\alpha, \beta, \gamma > 0$ are all real constants and $\beta + \gamma > 1 + 2\alpha$, $\gamma > 1 + 2\alpha$. Then T has a unique fixed point.

Proof: Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}, n \in \mathbb{N}$ as follows and $Tx_n = x_{n-1}$ for $n=1, 2, 3, \dots$. Then, using (1) we obtain

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\geq \beta \left(\frac{d(x_{n+1}, Tx_{n+1}) d(x_{n+2}, Tx_{n+2})}{d(x_{n+1}, x_{n+2})} \right) + \gamma d(x_{n+1}, x_{n+2}) \dots\dots(1) \\
d(x_n, x_{n+1}) + \alpha \left(\frac{d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})}{1 + d(x_{n+2}, x_{n+1}) d(x_{n+1}, x_{n+1})} \right) \\
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d(x_n, x_{n+1}) + \alpha d(x_{n+2}, x_n) \geq \beta d(x_{n+1}, x_n) + \gamma d(x_{n+1}, x_{n+2}) \\
(1 + \alpha - \beta) d(x_n, x_{n+1}) \geq (-\alpha + \gamma) d(x_{n+1}, x_{n+2})
\end{aligned}$$

$$d(x_{n+1}, x_{n+2}) \leq \left(\frac{1+\alpha-\beta}{-\alpha+\gamma} \right) d(x_n, x_{n+1})$$

$$d(x_{n+1}, x_{n+2}) \leq h d(x_n, x_{n+1}) \dots (2) \text{ where } h = \left(\frac{1+\alpha-\beta}{-\alpha+\gamma} \right) < 1$$

In the same way, we have $d(x_{n+1}, x_{n+2}) \leq h d(x_n, x_{n+1})$.

By (2), we get $d(x_{n+1}, x_{n+2}) \leq h^2 d(x_{n-1}, x_n)$. Continue this process, we get in general $d(x_n, x_{n+1}) \leq h^{n+1} d(x_1, x_0)$. Since $0 \leq h < 1$ as $n \rightarrow \infty$, $h^{n+1} \rightarrow 0$. Hence $\{x_n\}$ is a d-cauchy sequence in X . Thus $\{x_n\}$ dislocated converges to some u in X . Since T is continuous we have $T(u) = \lim T(x_n) = \lim x_{n+1} = u$. Thus u is a fixed point T .

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$$d(x^*, y^*) + \alpha \left(\frac{d(y^*, x^*) + d(x^*, y^*)}{1 + d(y^*, y^*) d(x^*, y^*)} \right) \geq \beta \left(\frac{d(x^*, x^*) d(y^*, y^*)}{d(x^*, y^*)} \right) + \gamma d(x^*, y^*)$$

$$d(x^*, y^*) + 2\alpha d(x^*, y^*) \geq \gamma d(x^*, y^*)$$

$$d(x^*, y^*) \leq \left(\frac{1+2\alpha}{\gamma} \right) d(x^*, y^*)$$

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$$d(x_n, x_{n+1}) + \alpha \left(\frac{d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})}{1 + d(x_{n+2}, x_{n+1}) d(x_{n+1}, x_{n+1})} \right) \\ \geq \beta \left(\frac{d(x_{n+1}, x_{n+1}) d(x_{n+2}, x_{n+1})}{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+1})} \right) + \gamma d(x_{n+1}, x_{n+2})$$

$$d(x_n, x_{n+1}) + \alpha d(x_{n+2}, x_n) \geq \gamma d(x_{n+1}, x_{n+2}) \\ (1 + \alpha) d(x_n, x_{n+1}) \geq (-\alpha + \gamma) d(x_{n+1}, x_{n+2})$$

$$d(x_{n+1}, x_{n+2}) \leq \left(\frac{1 + \alpha}{-\alpha + \gamma} \right) d(x_n, x_{n+1})$$

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Theorem 3.4 : Let (X, d) be a complete dislocated metric space. Let T be a continuous mapping from X to X satisfying the following condition:

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for all

$x, y \in X, x \neq y$, where $\alpha, \beta, \gamma > 0$ are all real constants and $\gamma > 1 + 2\alpha + \beta, \gamma > 1 + 2\alpha$.

Then T has a unique fixed point.

Proof: Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}, n \in \mathbb{N}$ as follows and $Tx_n = x_{n+1}$ for $n=1, 2, 3, \dots$. Then, using (1) we obtain

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$$\begin{aligned} d(x_n, x_{n+1}) + \alpha \left(\frac{d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})}{1 + d(x_{n+2}, x_{n+1}) d(x_{n+1}, x_{n+1})} \right) \\ \geq \beta \left(\frac{d(x_{n+1}, x_n)[1 + d(x_{n+2}, x_{n+1})]}{1 + d(x_{n+1}, x_{n+2})} \right) + \gamma d(x_{n+1}, x_{n+2}) \end{aligned}$$

$$d(x_n, x_{n+1}) + \alpha d(x_{n+2}, x_n) \geq \beta d(x_n, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})$$

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d(x^*, y^*) + \alpha \left(\frac{d(y^*, x^*) + d(x^*, y^*)}{1 + d(y^*, y^*) d(x^*, y^*)} \right) &\geq \beta \left(\frac{d(x^*, x^*) [1 + d(y^*, y^*)]}{1 + d(x^*, y^*)} \right) + \gamma d(x^*, y^*) \\
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