A Lower Bound for $\tau(n)$ of Any $k$-Perfect Numbers

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Abstract

A natural number $n$ is said to be $k$-perfect number if $\sigma(n) = k \cdot n$ for some integer $k \geq 2$. In this paper, we will provide a lower bound for $\tau(n)$ of any $k$-perfect numbers. The lower bound for $\tau(n)$ will help in determining if the number is a possible $k$-perfect or not. For example, for all $n$ where $\tau(n) < 40,427,833,596$, the number $n$ can never be a $k$-perfect number with $k \geq 25$.

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1 Introduction

The number of divisors of any natural number $n$, $n = \prod_{i=1}^{m} p_i^{\alpha_i}$, is given by the formula $\tau(n) = \sum_{d|n} 1 = \prod_{i=1}^{m} (\alpha_i + 1)$ and the sum of the divisors of any natural number $n$ is defined as $\sigma(n) = \sum_{d|n} d$ where $d$ is a divisor of $n$. In studying perfect numbers, the abundancy index, denoted by $I(n)$, is defined as $I(n) = \frac{\sigma(n)}{n}$. And lastly, a natural number $n$ is said to be $k$-perfect number if $\sigma(n) = k \cdot n$ for some integer $k \geq 2$. If $n$ is $k$-perfect, then $I(n) = \sum_{d|n} \frac{1}{d} = k$. On the other hand, we know that the $n$th harmonic number denoted by $H_n$ is defined as $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Clearly, $I(n) \leq H_n$ for all natural numbers $n$. 
2 Preliminary Results

Let us first consider some lemmas.

**Lemma 2.1.** For \( k \in \mathbb{N} \), the inequality
\[
\left(1 + \frac{1}{k(k+2)}\right)^k \leq 1 + \frac{1}{k+1} \leq \left(1 + \frac{1}{k(k+1)}\right)^k
\]
holds.

**Proof 2.2.** Consider first the inequality
\[
1 + \frac{1}{k+1} \leq \left(1 + \frac{1}{k(k+1)}\right)^k
\]
By binomial expansion on the RHS of the inequality, we have
\[
\left(1 + \frac{1}{k(k+1)}\right)^k = \sum_{i=0}^{k} \binom{k}{i} 1^{k-i} \left(\frac{1}{k(k+1)}\right)^i = 1 + \frac{1}{k+1} + \sum_{i=2}^{k} \binom{k}{i} 1^{k-i} \left(\frac{1}{k(k+1)}\right)^i.
\]
Clearly,
\[
0 \leq \sum_{i=2}^{k} \binom{k}{i} 1^{k-i} \left(\frac{1}{k(k+1)}\right)^i.
\]
Adding both sides by \( 1 + \frac{1}{k+1} \), we arrive on the desired inequality. On the other hand, consider the inequality
\[
\left(1 + \frac{1}{k(k+2)}\right)^k \leq 1 + \frac{1}{k+1}
\]
Raising both sides by \( k+2 \), we get
\[
\left(1 + \frac{1}{k(k+2)}\right)^{k+2} \leq \left(1 + \frac{1}{k+1}\right)^{k+2} \Leftrightarrow \left(1 + \frac{1}{x}\right)^x \leq \left(1 + \frac{1}{y}\right)^{y+1}.
\]
Since the
\[
\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \text{and} \quad \lim_{y \to +\infty} \left(1 + \frac{1}{y}\right)^{y+1} = e
\]
from below and from above respectively, by using the binomial theorem and the power series expansion of \( e \), then that proves the inequality.

**Lemma 2.3.** The inequality
\[
\sum_{i=2}^{n} \frac{1}{i} < \int_{1}^{n} \frac{1}{x} \, dx < \sum_{i=1}^{n} \frac{1}{i}
\]
holds.
Proof 2.4. By lemma 1,
\[
\left(1 + \frac{1}{k(k + 2)}\right)^k \leq 1 + \frac{1}{k + 1}
\]

By some manipulations,
\[
\left(\frac{(k + 1)(k + 1)}{k(k + 2)}\right)^k \leq \frac{k + 2}{k + 1} \Rightarrow \left(\frac{k + 1}{k}\right)^k \leq \left(\frac{k + 2}{k + 1}\right)^k \Rightarrow \left(\frac{k + 2}{k + 1}\right)^k = \left(\frac{k + 2}{k + 1}\right)^{k+1}.
\]

Thus, we get
\[
\left(1 + \frac{1}{k}\right)^k \leq \left(1 + \frac{1}{k + 1}\right)^{k+1} < e.
\]

Now, we consider the inequality
\[
\left(1 + \frac{1}{k}\right)^k < e \Rightarrow e^{\ln\left(1 + \frac{1}{k}\right)} < e^{\frac{1}{k}} \Rightarrow \ln\left(\frac{k + 1}{k}\right) < \frac{1}{k}.
\]

Therefore,
\[
\ln\left(\frac{k + 1}{k}\right) < \frac{1}{k} \Rightarrow \frac{k + 1}{k} > e^{\frac{1}{k}} \Rightarrow \frac{k + 1}{k} > e^{\frac{1}{k}} \Rightarrow \ln\left(\frac{k + 1}{k}\right) < \frac{1}{k}.
\]

The other inequality is left as an exercise.

Remark 2.5. In fact, the inequality
\[
\sum_{i=2}^{n} \frac{1}{i} < \int_{1}^{n} \frac{1}{x} \, dx
\]

appeared as a problem in the book of Rosen. This was noted here to be able to make the following connections.

The previous lemma can be written as
\[
H_n - 1 < H_n - (\gamma + \epsilon) < H_n
\]
where $\gamma$ is the Euler-Mascheroni constant, defined as

$$\gamma = \lim_{n \to +\infty} (H_n - \ln(n))$$

and $\epsilon$ can be seen as the error term. For more details about Euler-Mascheroni constant, you may look at the paper of Lagarias. From this inequality, we can have a bound for $\gamma$.

$$-\epsilon < \gamma < 1 - \epsilon$$

As $n \to +\infty$, $\epsilon \to 0$ and that will give us $0 < \gamma < 1$. In fact, $\gamma = 0.577215664901532860606512\ldots$ (see Sloane’s A001620 at OEIS.org)

3 Results and Discussion

**Theorem 3.1.** For positive integers $k_i$,

$$\sum_{i=1}^{n} \frac{1}{k_i} \leq \sum_{i=1}^{n} \frac{1}{i}$$

where for every $k_i$ and $k_j$, $k_i \neq k_j$ and for all $k_i$ and $k_{i+1}$, $k_i < k_{i+1}$.

**Proof 3.2.** It should be noted that equality holds if $k_i = i$. Now suppose that there exists $k_i \neq i$. This would mean that in the set $S = \{1, 2, 3, \ldots, n\}$, there is $k_i \notin S$. Thus, $k_i > n$. Now, we have $k_i$’s such that

$$\frac{1}{k_i} < \frac{1}{n} < \frac{1}{j}$$

for all $j \in S$ such that $j \neq k_i$. Adding all unit fractions $\frac{1}{j}$ for $j \neq k_i$ and $j = k_i$, we get

$$\sum_{j \neq k_i} \frac{1}{k_i} + \sum_{j = k_i} \frac{1}{k_i} \leq \sum_{j \neq k_i} \frac{1}{j} + \sum_{j = k_i} \frac{1}{j}$$

and thus,

$$\sum_{i=1}^{n} \frac{1}{k_i} \leq \sum_{i=1}^{n} \frac{1}{i}$$

Suppose that $k_i$’s are not just any random natural numbers but rather all $k_i | n$ and the $n$ in the $\sum_{i=1}^{n} \frac{1}{k_i}$ will be replaced with $\tau(n)$. From this, we can rewrite the above inequality as

$$k = I(n) = \sum_{d | n} \frac{1}{d} = \sum_{d | \tau(n)} \frac{1}{d_i} \leq H_{\tau(n)}$$
Theorem 3.3 (A Lower bound of $\tau(n)$). For large $n$, $n$ can be a $k$-perfect number if the property

$$e^{k - \gamma} < \tau(n)$$

is satisfied.

Proof 3.4. It was already established that

$$k < H_{\tau(n)} = \ln(\tau(n)) + \gamma$$

Then,

$$k - \gamma < \ln(\tau(n)) \Rightarrow e^{k - \gamma} < \tau(n)$$

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References


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