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Bounded Continuous Linear Operator on Cone Normed Space $C'[a,b]$ to $C[a,b]$

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Abstract

In this paper we study boundedness and continuity of linear operator on cone normed space by showing that a linear mapping from cone normed space $C'[a,b]$ to $C[a,b]$ is continuous and bounded. Moreover, we study a set of bounded linear operator on cone normed space which is a cone Banach space. By means, we show that the cone Banach space is complete with its cone norm.

Keywords: Banach space, cone normed space, normed space, linear operator

1 Introduction

Functional analysis is a branch of mathematical analysis which studies vectors that are adjusted with several functions such as norm, operator, inner product, etc. In functional analysis we discuss metric spaces, norm spaces, inner product spaces, Banach spaces, and Hilbert spaces.
Now, some authors introduced generalization of metric spaces and normed spaces. In [1], the author introduced the concept of cone metric spaces and disscussed fixed point theorem of contractive mappings. Similarly, in [2] we found that a concept of cone normed space is introduced by the authors. Actually, cone normed space is a generalization of normed space. The difference between them is on their codomains. The real numbers $\mathbb{R}$ is used as a codomain on normed space while any Banach space E is used as a codomain on cone normed space. From [3] we know that real numbers $\mathbb{R}$ is a Banach space so we can conclude that cone normed space is a generalization of normed space. Some researches of typical cone normed space has been found in [4] and [5]. Darmawan [4] constructed a cone

normed function $C[a,b]$ valued on ℓ^p. On the other research, Murti in [5] studied about some fixed theorems of contractive mappings on $\mathbb{R}$ cone normed space $\mathbb{R}^2$ valued.

Regularly, when we study about normed spaces some characteristics of normed spaces like a linear operator will be disscussed too. Unfortunately, researches about linear operator on cone normed space is insufficient. Therefore, the authors of this paper will disscuss about this topic. Especially, a linear operator on cone normed space $C'[a,b]$ to $C[a,b]$. We discuss about its boundedness and continuity. Moreover, we will show that a bounded linear operator space on cone normed space $C'[a,b]$ to $C[a,b]$ is a cone Banach space.

2 Preliminary Notes

In this section we review some theories in cone set and cone normed spaces. Before talking about a cone, the authors assume that the readers have studied normed spaces and bounded continuous linear operator on normed space. It means several vector spaces such as ℓ^1, $C[a,b]$, and $C'[a,b]$ are Banach spaces with their standard norms.

Definition 2.1 [2] *Let E be a real Banach space and P_E is a subset of E. Then P_E is called a cone if satisfies the following conditions*:
(C1) *P_E is closed, $P_E \neq \{\mathbf{0}_E\}$, $P_E \neq \emptyset$*
(C2) *$\alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0$ and $x, y \in P_E \Rightarrow \alpha x + \beta y \in P_E$*
(C3) *$P_E \cap -P_E = \{\mathbf{0}_E\}$.*

For a given cone $P_E \subseteq E$, we can define a partial ordering " $\preccurlyeq$ " with respect to P_E by $x \preccurlyeq y$ if and only if $y - x \in P_E$. In this paper we write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in intP_E$, where $intP_E$ denoted the interior of P_E. The cone P_E is normal if there is a number $K > 0$ such that for all $x, y \in E, \mathbf{0}_E \preccurlyeq x \preccurlyeq y$ then $\|x\|_E \leq K\|y\|_E$. The least positive number satisfying this condition is called the normal constant [1,2]. Through this paper, we assume that E is a real Banach space.

Definition 2.2 [4] *Let E be a real Banach space, $A \subseteq E$, and P_E is a cone. The set A is upper bounded if there is $t \in E, t \succcurlyeq \mathbf{0}_E$ such that $a \preccurlyeq t, \forall a \in A$. Then the element t is an upper bound for A. Moreover, an upper bound $t' \in E$ is a supremum (least upper bound) of A if $t' \preccurlyeq t$ for any upper bound $t \in A$. We denote t' as supA. The cone P_E has a supremum property if for every upper bounded set A in P_E least upper bound exists in P_E.*

Lemma 2.3 [4] *Let E be a Banach space, $P_E \subseteq E$, and P_E is a cone. Given a sequence (yn) in E that converges to $y \in E$. If $w \in E$ and $y_n \preccurlyeq w$, for $n \in \mathbb{N}$ then $y \preccurlyeq w$.*

Lemma 2.4 [4] *Let E be a Banach space, $P_E \subseteq E$, and P_E is a cone. For any $x, y \in P_E$ and $\alpha \in \mathbb{R}, \alpha \geq 0$ then*
(a) *if $x \ll y \Rightarrow x \preccurlyeq y$.*
(b) *If $x \preccurlyeq y \Rightarrow \alpha x \preccurlyeq \alpha y$.*

Definition 2.5 [2] *Let X be a real vector space and E be a real Banach space. Suppose the function $\|.\|_X^E : X \to E$ satisfies:*
(NC1) $\|x\|_X^E \succcurlyeq \mathbf{0}_E, \forall x \in X$ *and* $\|x\|_X^E = \mathbf{0}_E \Leftrightarrow x = \mathbf{0}_X$
(NC2) $\|\alpha x\|_X^E = |\alpha| \|x\|_X^E, \forall x \in X$ *and* $\alpha \in \mathbb{R}$
(NC3) $\|x + y\|_X^E \preccurlyeq \|x\|_X^E + \|y\|_X^E, \forall x, y \in X$.
Then $\|.\|_X^E$ is called a cone norm on X and $(X, \|.\|_X^E)$ is called a cone normed space.

Definition 2.6 [2] *Let E be a Banach space, $(X, \|.\|_X^E)$ be a cone normed space, $x \in X$, and (x_n) is a sequence in X. The sequence (x_n) converges to x whenever $\forall c \in E$ with $\mathbf{0}_E \ll c$, there is a natural number N such that $\|x_n - x\|_X^E \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x \;\; n \to \infty$.*

Definition 2.7 [2] *Let E be a Banach space, $(X, \|.\|_X^E)$ be a cone normed space and (x_n) is a sequence in X. The sequence (x_n) is said to be a Cauchy sequence whenever $\forall c \in E$ with $\mathbf{0}_E \ll c$, there is a natural number N such that $\|x_n - x_m\|_X^E \ll c$ for all $n, m \geq N$.*

Definition 2.8 [2] *A cone normed space $(X, \|.\|_X^E)$ is said to be a cone Banach space if any Cauchy sequence in X converges.*

Definition 2.9 [2] *Let $(X, \|.\|_X^E)$ be a cone normed space. A subset A in X is cone bounded if $\{\|x\|_X^E : x \in A\}$ upper bounded.*

Theorem 2.10 [4] *Let $(X, \|.\|_X^E)$ be a cone normed space and P_E is a normal cone with normal constant K. If (x_n) is a sequence in X and $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \|x_n\|_X^E = \|x\|_X^E$.*

Theorem 2.11 [4] *Let $(X, \|.\|_X^E)$ be a cone normed space and P_E be a cone of Banach space E. Suppose that (x_n) and (y_n) are sequences in X which converge to $x, y \in X$ respectively and let (α_n) be a sequence in $\mathbb{R}$ which converges to α in $\mathbb{R}$. Then*
(a) $\lim_{n\to\infty} (x_n + y_n) = x + y$,
(b) $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

Definition 2.12 [2] *Let $(X, \|.\|_X^E)$ and $(Y, \|.\|_Y^E)$ be cone normed spaces. Suppose T is a linear operator from X to Y. Then*
(a) *T is continuous for some fixed points $x_0 \in X$ if given $c \in E, c \gg \mathbf{0}_E$ there is $t \in E, t \gg \mathbf{0}_E$ such that $\|T(x) - T(x_0)\|_X^E \ll c$ whenever $x \in X$ and*

$\|x - x_0\|_X^E \ll t$

(b) *T is continuous on if T is continuous at every point of X*

(c) *T is uniformly continuous if given $c \in E, c \gg \mathbf{0}_E$ there is $t \in E$, $t \gg \mathbf{0}_E$ such that $\|T(x) - T(x_0)\|_X^E \ll c$ whenever $x, u \in X$ and $\|x - u\|_X^E \ll t$*

(d) *T is sequentially continuous if given any sequence (x_n) in X which converges to $x_0 \in X$ then the sequence $(T(x_n))$ in Y converges to $T(x_0) \in Y$.*

According to Definition 2.12 we have

Theorem 2.13 [2] *Let $(X, \|.\|_X^E)$ and $(Y, \|.\|_Y^E)$ be cone normed spaces. Suppose T is a linear operator from X to Y. The following are equivalent*

(a) T is uniformly continuous on X

(b) T is continuous on X

(c) T is continuous at fixed point $x_0 \in X$

(d) T is continuous at $\mathbf{0}_X \in X$

(e) T is sequentially continuous on X

Definition 2.14 [2] *Let $(X, \|.\|_X^E)$ and $(Y, \|.\|_Y^E)$ be cone normed spaces and let $T: X \to Y$ is a linear operator from X to Y. A linear operator T is cone bounded if the set $T(B_{t^*}(\mathbf{0}_X))$ is a cone bounded set, for fixed t^* with $t^* \in E$ and $t^* \gg \mathbf{0}_E$.*

Definition 2.15 [2] *Let T be a linear operator from cone normed space $(X, \|.\|_X^E)$ to cone normed space $(Y, \|.\|_Y^E)$. Given a fixed $t^* \in E$ with $t^* \gg \mathbf{0}_E$. T is cone bounded if there is $t \in E, t \succcurlyeq \mathbf{0}_E$ such that $\|T(x)\|_X^E \preccurlyeq t$, $\forall x \in X$ whenever $\|x\|_X^E \ll t^*$.*

3 Main Results

Our goal is to study the boundedness and continuity of linear operator on cone normed space. According to [2] we can establish a cone norm on any real normed space X by defining the cone norm as

$$\|x\|_X^E := \left(\frac{\|x\|_X}{2}, \frac{\|x\|_X}{2^2}, \dots\right) = \left(\frac{\|x\|_X}{2^n}\right),$$

for all $x \in X$. In this paper, the authors choose cone normed space $C[a,b]$ and cone normed space $C'[a,b]$ because of its continuity. The cone of ℓ^1 is P_{ℓ^1} that definded by

$$P_{\ell^1} := \{\bar{x} \in \ell^1 : \bar{x} = (x_i) = (x_1, x_2, \dots), x_i \geq 0, \forall i \in \mathbb{N}\}$$

and the interior of P_{ℓ^1} is

$$intP_{\ell^1} := \{\bar{x} \in \ell^1 : \bar{x} = (x_i), x_i > 0, \forall i \in \mathbb{N}\}.$$

Theorem 3.1 *Let $C[a,b]$ be a normed space with supremum norm $\|.\|_\infty$ and a function $\|.\|_C^{\ell^1}: C[a,b] \to \ell^1$ defined by*

$$\|f\|_C^{\ell^1} := \left(\frac{\|f\|_\infty}{2}, \frac{\|f\|_\infty}{2^2}, \dots\right) = \left(\frac{\|f\|_\infty}{2^n}\right)$$

for any $f \in C[a,b]$ *and* $n \in \mathbb{N}$. *A function* $\|.\|_C^{\ell^1}$ *is a cone normed on* $C[a,b]$ *and* $(C[a,b], \|.\|_C^{\ell^1})$ *is cone normed space* $C[a,b]$ *with* ℓ^1 *valued or just cone normed space* $C[a,b]$.

By using the convergent and Cauchy properties on normed space $C[a,b]$ we have a relationship between normed space $C[a,b]$ and cone normed space $C[a,b]$.

Proposition 3.2 *Let* $C[a,b]$ *be a cone normed space and* $f \in C[a,b]$. *If* (f_n) *is a sequence in* $C[a,b]$ *then*

(a) *The sequence* (f_n) *in cone normed space* $C[a,b]$ *converges to* $f \in C[a,b]$ *if and only if the sequence* (f_n) *in normed space* $C[a,b$ *converges to* f.

(b) *The sequence* (f_n) *in cone normed space* $C[a,b]$ *is a Cauchy sequence if and only if the sequence* (f_n) *is a Cauchy sequence in normed space* $C[a,b]$.

Corollary 3.3 *Cone normed space* $C[a,b]$ *is a cone Banach space.*

Similarly with cone normed space $C[a,b]$, we have

Theorem 3.4 *Let* $C'[a,b]$ *be a normed space with* $\|.\|_{C'}$ *and a function* $\|.\|_{C'}^{\ell^1}: C'[a,b] \to \ell^1$ *defined by*

$$\|g\|_{C'}^{\ell^1} := \left(\frac{\|g\|_{C'}}{2}, \frac{\|g\|_{C'}}{2^2}, \dots\right) = \left(\frac{\|g\|_{C'}}{2^n}\right)$$

for any $g \in C'[a,b]$ *and* $n \in \mathbb{N}$. *A function* $\|.\|_{C'}^{\ell^1}$ *is a cone normed on* $C'[a,b]$ *and* $(C'[a,b], \|.\|_{C'}^{\ell^1})$ *is cone normed space* $C'[a,b]$ *with* ℓ^1 *valued or just cone normed space* $C'[a,b]$.

Proposition 3.5 *Let* $C'[a,b]$ *be a cone normed space and* $g \in C'[a,b]$. *If* (g_n) *is a sequence in* $C'[a,b]$ *then*

(a) *The sequence* (g_n) *in cone normed space* $C['a,b]$ *converges to* $g \in C'[a,b]$ *if and only if the sequence* (g_n) *in normed space* $C'[a,b]$ *converges to* g.

(b) *The sequence* (g_n) *in cone normed space* $C'[a,b]$ *is a Cauchy sequence if and only if the sequence* (g_n) *is a Cauchy sequence in normed space* $C'[a,b]$.

Corollary 3.6 *Cone normed space* $C'[a,b]$ *is a cone Banach space.*

Now, let us to establish a linear operator from $(C'[a,b], \|.\|_{C'}^{\ell^1})$ to $(C[a,b], \|.\|_C^{\ell^1})$.

Proposition 3.7. *Let* $(C'[a,b], \|.\|_{C'}^{\ell^1})$ *and* $(C[a,b], \|.\|_C^{\ell^1})$ *be cone normed space and any fixed function* $\mathcal{H} \in C[a,b]$. *Suppose that* $T_{\mathcal{H}}: C'[a,b] \to C[a,b]$ *is defined by*

$$T_{\mathcal{H}}(f) = \mathcal{H}f', \ \forall f \in C'[a,b].$$

$T_{\mathcal{H}}$ *is continuous linear operator on cone normed space and cone bounded* $C'[a,b]$.

Proof. We left it to the readers to prove that $T_{\mathcal{H}}$is a linear operator. Given any $c \in \ell^1$ with $c \gg \bar{0}$, we can find $t \in \ell^1$, $t \gg \bar{0}$ such that for $f, g \in C'[a,b]$ and $\|f-g\|_{C'}^{\ell^1} \ll t$ then $\|T_{\mathcal{H}}(f) - T_{\mathcal{H}}(g)\|_C^{\ell^1} \ll c$.On the other hand,

$$\|T_{\mathcal{H}}(f) - T_{\mathcal{H}}(g)\|_C^{\ell^1} = \|T_{\mathcal{H}}(f-g)\|_C^{\ell^1} = \|\mathcal{H}(f-g)'\|_C^{\ell^1} = \|\mathcal{H}(f'-g')\|_C^{\ell^1}$$
$$= \left(\frac{\|\mathcal{H}(f'-g')\|_\infty}{2^n}\right) = \left(\frac{\sup_{t\in[a,b]} |\mathcal{H}(t)(f'(t)-g'(t))|}{2^n}\right).$$

Since

$$\sup_{t\in[a,b]} \left|\mathcal{H}(t)(f'(t)-g'(t))\right| \leq \sup_{t\in[a,b]} |\mathcal{H}(t)| \sup_{t\in[a,b]} \left|f'(t)-g'(t)\right|$$

or we can say that

$$\left\|\mathcal{H}(f'-g')\right\|_\infty \leq \|\mathcal{H}\|_\infty \left\|f'-g'\right\|_\infty.$$

Hence for all $n \in \mathbb{N}$ we have

$$\left(\frac{\|\mathcal{H}(f'-g')\|_\infty}{2^n}\right) \leq \|\mathcal{H}\|_\infty \left(\frac{\|f'-g'\|_\infty}{2^n}\right).$$

It means

$$\|\mathcal{H}\|_\infty \left(\frac{\|f'-g'\|_\infty}{2^n}\right) - \left(\frac{\|\mathcal{H}(f'-g')\|_\infty}{2^n}\right) \geq 0 \text{ or } \|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1} - \|\mathcal{H}(f'-g')\|_C^{\ell^1} \geq 0.$$

Since $\|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1} - \|\mathcal{H}(f'-g')\|_C^{\ell^1} \in \ell^1$ and $\|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1} - \|\mathcal{H}(f'-g')\|_C^{\ell^1} \geq 0$ then $\|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1} - \|\mathcal{H}(f'-g')\|_C^{\ell^1} \in P_{\ell^1}$. By our notations we have

$$\|\mathcal{H}(f'-g')\|_C^{\ell^1} \preccurlyeq \|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1}. \tag{1}$$

Similarly, we can show that

$$\|f'-g'\|_C^{\ell^1} \preccurlyeq \|f-g\|_{C'}^{\ell^1}. \tag{2}$$

By (1), (2), and Lemma 2.4 (b) we have

$\|T_{\mathcal{H}}(f) - T_{\mathcal{H}}(g)\|_C^{\ell^1} = \|\mathcal{H}(f'-g')\|_C^{\ell^1} \preccurlyeq \|\mathcal{H}\|_\infty \|f'-g'\|_C^{\ell^1} \ll t.$

So, we can choose $\mathrm{t} := \frac{\mathrm{c}}{\|\mathcal{H}\|_\infty}$ such that $\|\mathrm{T}_{\mathcal{H}}(\mathrm{f}) - \mathrm{T}_{\mathcal{H}}(\mathrm{g})\|_{\mathrm{C}}^{\ell^1} \ll \mathrm{c}$ for $\mathrm{f}, \mathrm{g} \in \mathrm{C}'[\mathrm{a}, \mathrm{b}]$. This means $\mathrm{T}_{\mathcal{H}}$ is a uniformly continuous linear operator on $(\mathrm{C}'[\mathrm{a}, \mathrm{b}], \|.\|_{\mathrm{C}'}^{\ell^1}$. By Theorem 2.13, a linear operator $\mathrm{T}_{\mathcal{H}}$ is continuous on $(\mathrm{C}'[\mathrm{a}, \mathrm{b}], \|.\|_{\mathrm{C}'}^{\ell^1})$.

Next, we show that $T_{\mathcal{H}}$ is cone bounded. By Theorem 2.13, we have $T_{\mathcal{H}}$ is continuous at $\theta \in C'[a,b]$ for a fixed $t \in \ell^1$ with $t \gg \bar{0}$ we can choose $w \in \ell^1, w \succcurlyeq \bar{0}$ which satisfies $w := \|\mathcal{H}\|_\infty t$ such that $\forall f \in C'[a,b]$ and $\|f-\theta\|_C^{\ell^1} = \|f\|_C^{\ell^1}$ then $\|T_{\mathcal{H}}(f) - T_{\mathcal{H}}(\theta)\|_C^{\ell^1} = \|T_{\mathcal{H}}(f) - \theta\|_C^{\ell^1} = \|T_{\mathcal{H}}(f)\|_C^{\ell^1} = \|\mathcal{H}f'\|_C^{\ell^1}$

$$= \|T_{\mathcal{H}}(f)\|_C^{\ell^1} = \|\mathcal{H}f'\|_C^{\ell^1} \preccurlyeq \|\mathcal{H}\|_\infty \|f'\|_C^{\ell^1}$$

$$\preccurlyeq \|\mathcal{H}\|_\infty \|f\|_{C'}^{\ell^1} \ll \|\mathcal{H}\|_\infty t.$$

By Lemma 2.4 (a), we have $\|\mathcal{H}\|_\infty \|f\|_{C'}^{\ell^1} \ll \|\mathcal{H}\|_\infty t$ consequences $\|\mathcal{H}\|_\infty \|f\|_{C'}^{\ell^1} \preccurlyeq \|\mathcal{H}\|_\infty t = w$. Therefore we have $\|T_{\mathcal{H}}(f) - T_{\mathcal{H}}(\theta)\|_C^{\ell^1} = \|T_{\mathcal{H}}(f)\|_C^{\ell^1} \preccurlyeq w$. This shows that $T_{\mathcal{H}}$ is cone bounded.

Proposition 3.8. *Let $\tilde{B}(C'[a,b], C[a,b])$ be a cone bounded linear operators from $(C'[a,b], \|.\|_{C'}^{\ell^1})$ to $(C[a,b], \|.\|_C^{\ell^1})$. Let a function $\|.\|_{\tilde{B}}^{\ell^1} : \tilde{B}(C'[a,b], C[a,b]) \to \ell^1$ defined by*

$$\|T\|_{\tilde{B}}^{\ell^1} := \sup\left\{\|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\}$$

for all $T \in \tilde{B}(C'[a,b], C[a,b])$. A function $\|.\|_{\tilde{B}}^{\ell^1}$ is a cone norm for $\tilde{B}(C'[a,b], C[a,b])$.

Proof. Before we show $\|.\|_{\tilde{B}}^{\ell^1}$ is a cone norm for $\tilde{B}(C'[a,b], C[a,b])$ we should prove $\tilde{B}(C'[a,b], C[a,b])$ is a real vector space. For our convenience we will left it to the readers.

(NC1) Clearly, $\|T\|_{\tilde{B}} \succcurlyeq \bar{0}$ since $\|.\|_C^{\ell^1}$ is a cone norm for $C[a,b]$ then $\sup\left\{\|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \succcurlyeq \bar{0}$ or we can say $\|T\|_{\tilde{B}}^{\ell^1} \succcurlyeq \bar{0}$.
Now, if given $\|T\|_{\tilde{B}}^{\ell^1} = \bar{0}$, it means

$$\sup\left\{\|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} = \bar{0}$$

Therefore we have $\|T\|_C^{\ell^1} = \bar{0}$ for $f \in C'[a,b]$ with $\|f\|_{C'}^{\ell^1} \ll t^*$. Consequently, we have $T(f) = \theta, \forall f \in C'[a,b]$. This shows $T(f) = \Theta(f), \forall f \in C'[a,b]$. We say $T = \Theta$. In this case θ is a null vector in $C'[a,b]$ and $C[a,b]$ while ?? is a null vector in $(C'[a,b], C[a,b])$ which defined by $\Theta(f) = \theta, \forall f \in C'[a,b]$ and $\theta(t) = 0, \forall t \in [a,b]$. Conversely, if $T = \Theta$ we get $T(f) = \Theta(f), \forall f \in C'[a,b]$ such that if $\|f\|_{C'}^{\ell^1} \ll t^*$ then $\|T(f)\|_C^{\ell^1} = \|\Theta(f)\|_C^{\ell^1} = \|\theta\|_C^{\ell^1} = \bar{0}$. Hence

$$\sup\left\{\|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} = \bar{0}$$

or we can say $\|T\|_{\tilde{B}}^{\ell^1} = \bar{0}$.

(NC2) By the properties of cone norm and supremum we have

$$\begin{aligned}\|\alpha T\|_{\tilde{B}}^{\ell^1} &= \sup\left\{\|\alpha T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &= \sup\left\{|\alpha| \|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &= |\alpha| \sup\left\{\|T(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &= |\alpha| \|T\|_{\tilde{B}}^{\ell^1}.\end{aligned}$$

(NC3) By the properties of cone norm and supremum we have

$$\begin{aligned}\|T_1 + T_2\|_{\tilde{B}}^{\ell^1} &= \sup\left\{\|(T_1 + T_2)(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &= \sup\left\{\|(T_1)(f) + T_2(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &\preccurlyeq \sup\left\{\|(T_1)(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} + \sup\left\{\|(T_2)(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \\ &= \|T_1\|_{\tilde{B}}^{\ell^1} + \|T_2\|_{\tilde{B}}^{\ell^1}.\end{aligned}$$

We see that $\|.\|_{\tilde{B}}^{\ell^1}$ satisfies (NC1)-(NC3) conditions. So, we can conclude that $\|.\|_{\tilde{B}}^{\ell^1}$ is a cone norm for $\tilde{B}(C'[a,b], C[a,b])$.

Proposition 3.9. *A cone normed space $\tilde{B}(C'[a,b], C[a,b])$ is a cone Banach space with a cone norm $\|.\|_{\tilde{B}}^{\ell^1}$.*

Proof. Let (T_n) be a Cauchy sequence in $\tilde{B}(C'[a,b], C[a,b])$, this means for given $\forall c \in \ell^1, c \gg \bar{0}, \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$ we have

$$\|T_n - T_m\|_{\tilde{B}}^{\ell^1} = \sup\left\{\|T_n(f) - T_m(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\} \ll c.$$

On the other hand, for all $n, m \in \mathbb{N}$ we have

$$\|T_n(f) - T_m(f)\|_C^{\ell^1} \preccurlyeq \sup\left\{\|T_n(f) - T_m(f)\|_C^{\ell^1} : f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*\right\}$$

then $\forall n, m \geq N$ we have $\|T_n(f) - T_m(f)\|_C^{\ell^1} \ll c, \forall f \in C'[a,b]$ (3)

It means for any $f \in C'[a,b]$, a sequence $(T_n(f))$ is a Cauchy sequence in $C[a,b]$. Since $C[a,b]$ is a cone Banach space then $(T_n(f))$ converges to a point in $C[a,b]$, $T_n(f) \to T(f)$ as $n \to \infty$. We will show that $T \in \tilde{B}(C'[a,b], C[a,b])$. Since $T_n \in \tilde{B}(C'[a,b], C[a,b]), \forall n \in \mathbb{N}$, by Theorem 2.11 (a) and (b), $\forall f, g \in C'[a,b]$, $\forall \alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned} T(\alpha f + \beta g) &= \lim_{n\to\infty} T_n(\alpha f + \beta g) = \lim_{n\to\infty}(\alpha T_n(f) + \beta T_n(g)) \\ &= \alpha \lim_{n\to\infty} T_n(f) + \beta \lim_{n\to\infty} T_n(g) = \alpha T(f) + \beta T(g). \end{aligned}$$

This means T is a linear operator from $C'[a,b]$ to $C[a,b]$. Since T_n is cone bounded $\forall n \in \mathbb{N}$, then $\exists t \in \ell^1$, $t \succcurlyeq \bar{0}$ such that $\|T_n(f)\|_C^{\ell^1} \preccurlyeq t, \forall f \in C'[a,b]$ with $\|f\|_{C'}^{\ell^1} \ll t^*$. By Lemma 2.3 and Theorem 2.10, for all $f \in C'[a,b]$ with the condition $\|f\|_{C'}^{\ell^1} \ll t^*$, we have

$$\|T_n(f)\|_C^{\ell^1} = \left\|\lim_{n\to\infty} T_n(f)\right\|_C^{\ell^1} = \lim_{n\to\infty}\|T_n(f)\|_C^{\ell^1} \preccurlyeq t.$$

This shows T is cone bounded. Therefore $T \in \tilde{B}(C'[a,b], C[a,b])$. Now, we will show that $T_n \to T$. Since (3) holds for all $m \geq N$ and $T_m(f) \to T(f)$ as $m \to \infty$, then for all $n \geq N$ we have

$$\|T_n(f) - T(f)\|_C^{\ell^1} = \left\|T_n(f) - \lim_{m\to\infty} T_m(f)\right\|_C^{\ell^1} = \lim_{n\to\infty}\|T_n(f) - T_m(f)\|_C^{\ell^1} \ll c.$$

By taking $f \in C'[a,b], \|f\|_{C'}^{\ell^1} \ll t^*$ and the supremum of $\|T_n(f) - T(f)\|_C^{\ell^1}$ we have

$$\|T_n(f) - T(f)\|_C^{\ell^1} \ll c.$$

So, $T_n \to T$ as $n \to \infty$. This shows that $\tilde{B}(C'[a,b], C[a,b])$ is a cone Banach space.

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