Applied Mathematical Sciences, Vol. 4, 2010, no. 23, 1099 - 1105

On Semiring of Intuitionstic Fuzzy Matrices

S. Sriram and P. Murugadas

Mathematics Section, FEAT
Annamalai University
Annamalainagar - 608 002, India
ssm_3096@yahoo.co.in
bodi_muruga@yahoo.com

Abstract

In this paper, we study the concept of semiring of intuitionistic fuzzy matrices(IFMs). We prove that the IFMs forms an intuitionistic fuzzy algebra and vector space over [0,1]. Some properties of IFMs are studied using the definition of comparability of IFMs.

Mathematics Subject Classification: Primary 05A15; Secondary 65F25

Keywords: Intuitionistic fuzzy matrix(IFM)

1. Introduction

Atanassov[1] generalized the notion of Zadeh's fuzzy set to the concept of intuitionistic fuzzy set (IFS), which is composed of membership degree, non-membership degree and hesitation degree of an element x in a set A. Im.et.al[7] defined the concept IFMs as a natural generalization of fuzzy matrices and they studied the determinant of square IFMs. Khan.S.K and Pal.M[6] studied some operations on IFMs. Jeong.N.G and Park.S.W[4] investigated the period of powers of square IFMs and gave some results for the equivalence IFMs and idempotent. Lee.H.Y and Jeong.N.G[5] decomposed a transtive IFM into sum of a nilpotent intuitionistic fuzzy matrix and a symmetic intuitionistic fuzzy matrix. They obtained a canonical form of the transitive IFM. In this paper, section 2 contains the preliminaries and some backgrounds in this study. We proved that F_n is an intuitionistic fuzzy algebra and form a vector space under component wise addition, component wise multiplication and scalar multiplication in section 3. In section 4 we proved intuitionistic fuzzy matrix multiplication is associative and distributive in F_n. Also, by using the definition of comparability of IFMs some properties are proved.

2. Preliminaries

Definition 2.1. *A* fuzzy matrix(FM) of order mxn is defined as $A = (a_{ij}, a_{ij\mu})$, where $a_{ij\mu}$ is the membership value of the element a_{ij} in$A$. Let F_{mn} denote the set of all fuzzy matrices of order mxn. If $m = n$, in short, we write F_n, the set of all square matrices of order n.

Definition 2.2[5]. An intuitionistic fuzzy matrix (IFM)denote it by F is a matrix of pairs $A = (< a_{ij}, a'_{ij} >)$ of a non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j. Let F_{mn} denote the set of all mxn intuitionistic fuzzy matrices. If $m = n$, in short, we write F_n.

Definition 2.3. Let a and b be two elements of an IFM F such that,

$a =< a_{ij}, a_{ij} >, \; b =< b_{ij}, b'_{ij} >$, then component wise addition and multiplication is defined as

$$a + b =< \max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\} >$$
$$a \bullet b =< \min\{a_{ij}, b_{ij}\}, \max\{a'_{ij}, b'_{ij}\} >$$

for our convenience, we say $\max\{a_{ij}, b_{ij}\} = a_{ij} + b_{ij}$ and $\min\{a_{ij}, b_{ij}\} = a_{ij}b_{ij}$.

Definition 2.4. Let $A, B \in \mathrm{F}_{mn}$ such that $A = (< a_{ij}, a'_{ij} >)$ and $B = (< b_{ij}, b'_{ij} >)$, then the matrix addition is given by

$$A + B = (< \max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\} >) \in \mathrm{F}_{mn}$$

For $A = (< a_{ij}, a'_{ij} >) \in \mathrm{F}_{mn}$ and $B = (< b_{ij}, b'_{ij} >) \in \mathrm{F}_{np}$, then the matrix multiplication is given by,

$AB = (< \max_k\{\min\{a_{ik}, b_{kj}\}\}, \min_k\{\max\{a'_{ik}, b'_{kj}\}\} >)$, where $k = 1$ to $n, i = 1$ to m and $j = 1$ to p.

We can write $\max\{\min\{a_{ik}, b_{kj}\}\} = \sum_{k=1}^{p} a_{ik}b_{kj}$, and

$\min\{\max\{a'_{ik}, b'_{kj}\}\} = \prod_{k=1}^{p} (a'_{ik} + b'_{kj})$. The product AB is defined if and only if the number of columns of A is the same as the number of rows of B, A and B are said to be conformable for multiplication.

Definition 2.5. The mxn zero intuitionistic fuzzy matrix $\mathbf{0}$ is the matrix all of whose entries are $(< 0, 1 >)$. The nxn identity matrix $\mathbf{I}_n$ is defined by $(< \delta_{ij}, \delta'_{ij} >)$ such that $\delta_{ij} = 1, \delta'_{ij} = 0$ if $i = j$ and $\delta_{ij} = 0, \delta'_{ij} = 1$ if $i \neq j$. The mxn universal matrix $\mathbf{J}$ is the matrix all of whose entries are $(< 1, 0 >)$.

Definition 2.6. Let $A = (< a_{ij}, \; a'_{ij} >) \in \mathrm{F}_{mn}$ and $c \in F$, then the intutionistic fuzzy scalar multiplication is defined as

$cA = (< \min\{c, a_{ij}\}, \max\{1-c, a'_{ij}\} >) \in \mathrm{F}_{mn}$. For the universal matrix $\mathbf{J}$, by definition

$$c\mathbf{J} = (< \min\{c, 1\}, \max\{1 - c, 0\} >) = (< c, 1 - c >).$$

Under component wise multiplication,

$$cJ \bullet A = (< \min\{c, a_{ij}\}, \max\{1 - c, a'_{ij}\} >) = cA.$$

Definition 2.7. Let $A, B \in \mathrm{F}_{mn}$ such that $A = (< a_{ij}, a'_{ij} >)$ and $B = (< b_{ij}, b'_{ij} >)$, then we write $A \leq B$ if, $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.

Example 1. $\mathbf{O} \leq A \leq \mathbf{J}$

Definition 2.8. A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix, if every row and column contains exactly one $< 1, 0 >$ and all other entries are $< 0, 1 >$. Let $\mathbf{P}_n$ be the set of all mxn such matrices in F_n. If $A \in \mathbf{P}_n$, then $AA^T = A^T A = \mathbf{I}_n$, A^T is the transpose of A.

3. Section

In this section we prove that F_n is an intuitionistic fuzzy algebra and form a vector space under the component wise addition, component wise multiplication and scalar multiplication.

Theorem 3.1. The set F_{mn} is an intuitionistic fuzzy algebra under component wise addition and multiplication operation $(+, \bullet)$.

Proof: Clearly, $A + \mathbf{O} = A$ and $A \bullet \mathbf{J} = A$ for all $A \in \mathrm{F}_{mn}$. Hence the zero matrix $\mathbf{O}$ is the additive identity and the universal matrix $\mathbf{J}$ is the multiplicative identity. Thus identity element relative to the operation + and $\bullet$ exist. Also, $A + \mathbf{J} = \mathbf{J}$ and $A \bullet \mathbf{O} = \mathbf{O}$. Hence universal bound exist for all $A \in \mathrm{F}_{mn}$. For $A = (< a_{ij}, a'_{ij} >), B = (< b_{ij}, b'_{ij} >)$ and $C = (< c_{ij}, c'_{ij} >) \in \mathrm{F}_{mn}$.

$$A + (B + C) = (< a_{ij}, a'_{ij} >) + (< \max\{b_{ij}, c_{ij}\}, \min\{b'_{ij}, c'_{ij}\} >)$$
$$= (< \max\{a_{ij}, b_{ij}, c_{ij}\}, \min\{a'_{ij}, b'_{ij}, c'_{ij}\} >) \tag{3.1}$$
$$\text{Also, } (A + B) + C = (< \max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\} >) + (< c_{ij}, c'_{ij} >)$$
$$= (< \max\{a_{ij}, b_{ij}, c_{ij}\}, \min\{a'_{ij}, b'_{ij}, c'_{ij}\} >) \tag{3.2}$$

from (3.1) and (3.2) $A + (B + C) = (A + B) + C$
similarly, we can prove $A \bullet (B \bullet C) = (A \bullet B) \bullet C$. Hence associativity law under + and $\bullet$ is satisfied.

$$\text{Further, } A + (A \bullet B) = (< a_{ij}, a'_{ij} >) + (< \min\{a_{ij}, b_{ij}\}, \max\{a'_{ij}, b'_{ij}\} >)$$
$$= (< \max\{a_{ij}, \min\{a_{ij}, b_{ij}\}\}, \min\{a'_{ij}, \max\{a'_{ij}, b'_{ij}\}\} >)$$
$$= (< a_{ij}, a'_{ij} >) = A$$

Similarly, $A \bullet (A + B) = A$. Therefore, condition for absorption is satisfied.

Assume $A \leq B$ or C

$$\begin{aligned} A \bullet (B+C) &= (<\min\{a_{ij}, \max\{b_{ij}, c_{ij}\}\}, \max\{a^{`}_{ij}, \min\{b^{`}_{ij}, c^{`}_{ij}\}\}>) \\ &= (<a_{ij}, a^{`}_{ij}>) = A \qquad (3.3) \end{aligned}$$

Also,

$$\begin{aligned} (A \bullet B) + (A \bullet C) &= (<\min\{a_{ij},, b_{ij}\}, \max\{a^{`}_{ij}, b^{`}_{ij}\}>) \\ &\quad + (<\min\{a_{ij}, c_{ij}\}, \max\{a^{`}_{ij}, c^{`}_{ij}\}>) \\ &= (<\max\{\min\{a_{ij}, b_{ij}\}, \min\{a_{ij}, c_{ij}\}\}, \min\{\max\{a^{`}_{ij}, b^{`}_{ij}\}, \\ &\quad \max\{a^{`}_{ij}, c^{`}_{ij}\}\}>) \\ &= (<a_{ij}, a^{`}_{ij}>) = A \qquad (3.4) \end{aligned}$$

from (3.3) and (3.4), $A \bullet (B+C) = (A \bullet B) + (A \bullet C)$
if, $A \geq B$ and C, then we have two cases $A \geq B \geq C$ or $A \geq C \geq B$
for, $A \geq B \geq C$ from (3.3) and (3.4) $A \bullet (B+C) = B = (A \bullet B) + (A \bullet C)$
$A \geq C \geq B$ from (3.3) and (3.4) $A \bullet (B+C) = C = (A \bullet B) + (A \bullet C)$
Therefore, $A \bullet (B+C) = (A \bullet B) + (A \bullet C)$
Similarly, we can prove $A \bullet (B+C) = (A \bullet B) + (A \bullet C)$. Thus the property of distributivity holds. Hence, F_{mn} is an intuitionistic fuzzy algebra under the operation $+$ and $\bullet$. □

Remark 3.1 F_{mn} is a commutative semiring with identity $\mathbf{O}$ and $\mathbf{J}$. □

Theorem 3.2. The set F_{mn} is an intuitionistic fuzzy vector space under the operations IFM addition and scalar multiplication.
Proof: For $A, B, C \in \mathrm{F}_{mn}$
Clearly, $A+B = B+A$ and $A+(B+C) = (A+B)+C$. Therefore commutative law and associative law holds in F_{mn}. Also, for all $A \in \mathrm{F}_{mn}$, there exist an element $\mathbf{O} \in \mathrm{F}_{\mathbf{mn}}$ such that $A + \mathbf{O} = \mathbf{A}$.

Again, for $c \in \mathbf{F}$

$$\begin{aligned} c(A+B) &= c\mathbf{J} \bullet (A+B) \quad \text{by definition 2.6} \\ &= c\mathbf{J} \bullet A + c\mathbf{J} \bullet B \quad \text{by theorem 3.1} \\ &= cA + cB. \end{aligned}$$

For $c_1, c_2 \in \mathbf{F}$

$$\begin{aligned} (c_1 + c_2)A &= (c_1 + c_2)\mathbf{J} \bullet A \\ &= (c_1\mathbf{J} + c_2\mathbf{J}) \bullet A \\ &= c_1\mathbf{J} \bullet \mathbf{A} + \mathbf{c_2 J} \bullet \mathbf{A} \\ &= c_1 A + c_2 A \end{aligned}$$

hence, F_{mn} is an intuitionistic vector space over $\mathbf{F}$. □

4. Section

In this section we prove matrix multiplication is associative and distributive in F_n.

Theorem 4.1. For any three IFMs A, B, C of order mxn, nxp, pxq respectively $(AB)C = A(BC)$.

Proof: Both $(AB)C$ and $A(BC)$ are defined and are of type mxq. Let $A = (< a_{ij}, a'_{ij} >), B = (< b_{jk}, b'_{jk} >)$ and $C = (< c_{kl}, c'_{kl} >)$ such that the ranges of the suffixes i, j, k and l are 1 to m , 1 to n,1 to p and 1 to q respectively. Now $(i, k)^{th}$ element of the product

$AB =< \sum_{j=1}^{n} a_{ij}b_{jk}, \prod_{j=1}^{n}(a'_{ij} + b'_{jk}) >$. The $(i, 1)^{th}$ element in the product $(AB)C$ is the sum of products of the corresponding elements in the i^{th} row of $AB, 1^{th}$ column of C with k common. Thus, $(i, 1)^{th}$ element of

$$(AB)C =< \sum_{k=1}^{p}(\sum_{j=1}^{n} a_{ij}b_{jk})c_{kl}, \prod_{k=1}^{p}(\prod_{j=1}^{n}(a'_{ij} + b'_{jk}) + c'_{kl}) >$$

$$=< \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kl}, \prod_{k=1}^{p}\prod_{j=1}^{n}(a'_{ij} + b'_{jk} + c'_{kl}) > \quad (4.1)$$

Now, $(j, 1)^{th}$ element of the product $BC =< \sum_{k=1}^{p} b_{jk}c_{kl}, \ \prod_{k=1}^{p}(b'_{jk} + c'_{kl}) >$.

Again the $(i, 1)^{th}$ element of the product $A(BC)$ is the sum of products of the corresponding elements in the i^{th} row of A and 1^{th} column of $BC.(i, l)^{th}$ element of

$$A(BC) =< \sum_{j=1}^{n} a_{ij}(\sum_{k=1}^{p} b_{jk}c_{kl}), \prod_{j=1}^{n}(a'_{ij} + \prod_{k=1}^{p}(b'_{jk} + c'_{kl})) >$$

$$=< \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}c_{kl}, \prod_{k=1}^{p}\prod_{j=1}^{n}(a'_{ij} + b'_{jk} + c'_{kl}) > \quad (4.2)$$

Hence from (4.1) and (4.2)$(AB)C = A(BC)$. □

Theorem 4.2. For any three matrices A, B and C of order mxn, nxp and nxp in F
$A(B + C) = AB + AC$.

Proof: Let $A = (< a_{ij}, a'_{ij} >), B = (< b_{jk}, b'_{jk} >)$ and $C = (< c_{jk}, c'_{jk} >)$ such that the ranges of suffixes i, j, k are $i = 1$ to $m, j = 1$ to n and $k = 1$ to p respectively. Now $(j, k)^{th}$ element of

$$B + C = (< \max\{b_{jk}, c_{jk}\}, \min\{b'_{jk}, c'_{jk}\} >)$$
$$= (< b_{jk} + c_{jk}, b'_{jk}c'_{jk} >)$$

$(i, k)^{th}$ element in the product of A and $(B + C)$, that is of $A(B + C)$ is the sum of the products of the corresponding elements in the i^{th} row A and k^{th} column of $B + C$

$$A(B + C) = (< \sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk}), \prod_{j=1}^{n}(a'_{ij} + b'_{jk}c'_{jk}) >) \quad (4.3)$$

Now, $(i,k)^{th}$ element of $(AB+AC)$ is

$$\begin{aligned} AB+AC &= (<\sum_{j=1}^{n} a_{ij}b_{jk}, \prod_{j=1}^{n}(a'_{ij}+b'_{jk})>) + (<\sum_{j=1}^{n} a_{jk}c_{jk}>, \prod_{j=1}^{n}(a'_{ij}+c'_{jk})>) \\ &= (<\sum_{j=1}^{n}(a_{ij}b_{jk}+a_{ij}c_{jk}), \prod_{j=1}^{n}(a'_{ij}+b'_{jk})\prod_{j=1}^{n}(a'_{ij}+c'_{jk})>) \\ &= (<\sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk}), \prod_{j=1}^{n}(a'_{ij}+b'_{jk}c'_{jk})>) \qquad (4.4) \end{aligned}$$

From (4.3) and (4.4) $A(B+C)=AB+AC$.

□

Theorem 4.3. Let $A,B \in \mathrm{F}_{mn}$, then $A \leq B$ iff $A+B=B$.
Proof: If $A \leq B$, then $A+B = (<\max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\}>)$

$= (<b_{ij}, b'_{ij}>) = B$ by definition 2.7.

Conversely, if $A+B=B$, then $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ this implies $A \leq B$.

□

Theorem 4.4. Let $A,B \in \mathrm{F}_{mn}$ if $A \leq B$ then for any $C \in \mathrm{F}_{np}, AC \leq BC$ and for any $D \in \mathrm{F}_{pm}, DA \leq DB$.

Proof. If $A \leq B$, then $a_{ik} \leq b_{ik}$ and $a'_{ik} \geq b'_{ik}$ for $i=1$ to m and $k=1$ to n. By fuzzy multiplication $a_{ik}c_{kj} \leq b_{ik}c_{kj}$ and $a'_{ik}c_{kj} \geq b'_{ik}c'_{kj}$ for $j=1$ to p. Again by fuzzy addition $\sum_{k=1}^{n} a_{ik}c_{kj} \leq \sum_{k=1}^{n} b_{ik}c_{kj}$ and $\sum_{k=1}^{n} a'_{ik}c'_{kj} \geq \sum_{k=1}^{n} b'_{ik}c'_{kj}$. Thus $AC \leq BC$. Similarly we can prove $DA \leq DA$. □

REFERENCES

[1] K. Atanassov, *Intuitionistic Fuzzy Sets,* Theory and Application, Physica-verlag, 1999.
[2] K. Atanassov, *Intuitionistic Fuzzy Sets,* Fuzzy Sets and Systems, **20**(1986), 87-96.
[3] M. Bhowmik and M. Pal, *Generalized Intuitionistic Fuzzy Matrices,* Far East Journal of Mathematical Sciences, **29(3)**(2008), 533-554.
[4] N. G. Jeong and S. W. Park, *The Equivalence Intuitionistic Fuzzy Matrix,*

For East Journal of Mathematical Sciences, **11(3)** (2003), 355-365.
[5] Lee. Hang. Youl and N. G. Jeong, *Canonical Form of Transitive Intuitionistic Fuzzy Matrices,* Honam Mathematical Journal, **27(4)** (2005), 543-550.
[6] S. K. Khan and M. Pal, *Some Operation on Intuitionistic Fuzzy Matrices,* Acta Ciencia Indica, XXXII M, 2006, 515-524.
[7] Young Bim Im, *The Determinant of Square Intuitionistic Fuzzy Matix,* Far East Journal of Mathematical Sciences, **3(5)**(2001), 789-796.
[8] L. A. Zadeh, *Fuzzy Sets,* Information and Control, **8**(1965), 338-353.

Received: November, 2009