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Deteriorating Inventory Model with Time Dependent Demand and Partial Backlogging

Vinod Kumar Mishra

Department of Computer Science & Engineering
Kumaon Engineering College
Dwarahat, Almora, - 263653, (Uttarakhand), India
vkmishra2005@gmail.com

Lal Sahab Singh

Department of Mathematics & Statistics
Dr.Ram Manohar Lohia Avadh University
Faizabad-224001, (Uttar Pradesh), India
singhdrlalsahab7@gmail.com

Abstract

In this paper, a deterministic inventory model is developed for deteriorating items in which shortages are allowed and partially backlogged. Deterioration rate is constant, Demand rate is linear function of time, backlogging rate is variable and is dependent on the length of the next replenishment. The model is solved analytically by minimizing the total inventory cost.

Mathematics Subject Classification: 90B05

Keywords: Inventory, deteriorating items, shortages, time dependent demand, partial backlogging.

1.0 Introduction

Deterioration is defined as decay, damage, spoilage evaporation and loss of utility of the product. Deterioration in inventory is a realistic feature and need to consideration it. Often we encounter products such as fruits, milk, drug, vegetables, and photographic films etc that have a defined period of life time. Such items are referred as deteriorating items. The loss due to deterioration cannot be avoided. Due to deterioration, inventory system faces the problem of shortages and loss of good will or loss of profit. Shortage is a fraction of those customers whose demand is not satisfied in the current period reacts to this by not returning the next period.

Inventory in deteriorating items first considered by Within [1957], he considered the deterioration of fashion goods at the end of prescribed storage period. In 1963 Ghare and Schrader extended the classical EOQ formula with exponential decay of inventory due to deterioration and gave a mathematical modeling of inventory in deteriorating items. Dave and Patel [1981] developed the first deteriorating inventory model with linear trend in demand. He considers demand as a linear function of time. Nahmias [1982] gave a review on perishable inventory theory. He reviewed the relevant literature on the problem of determining suitable ordering policies for both fixed life perishable inventory, and inventory subjected to continuous exponential decay. Rafaat [1991] gave a survey of literature on continuously deteriorating inventory models and he considered the effect of deterioration as a function of the on hand level of inventory. He focused to present an up-to-date and complete review of the literature for the continuously deteriorating mathematical inventory models. But all researchers assume that during shortage period all demand either backlogged or lost. In reality it is observed that some customers are willing to wait for the next replenishment. Abad [1996] considered this phenomenon in his model, optimal pricing and lot sizing under the conditions of perishable and partial backordering. He assumed that the backlogging rate depends upon the waiting time for the next replenishment. But he does not include the stock out cost (back order cost and lost sale cost). Chang and Dye [1999] developed an inventory model with time varying demand and partial backlogging. He considered that if longer the waiting time smaller the backlogging rate would be. So the proportion of the customer who would like to accept backlogging at time *t* is decreasing with the waiting time for the next replenishment. So to take care for this situation he defined a backlogging rate s. t.

$$B(t) = \frac{1}{1+\alpha\left(t_i - t\right)}$$

Where t_i is the time at which the i^{th} replenishment is making and α is backlogging parameter.

Goyal and Giri [2001] gave recent trends of modeling in deteriorating items inventory. They classified inventory models on the basis of demand variations and various other conditions or constraints. Ouyang, Wu and Cheng [2005] developed an inventory model for deteriorating items with exponential declining demand and partial backlogging. Dye, Hsieh and Ouyang [2007] find an optimal selling price and lot size with a varying rate of deterioration and exponential partial backlogging. They assume that a fraction of customers who backlog their orders increases exponentially as the waiting time for the next replenishment decreases. Shah and Shukla [2009] developed a deteriorating inventory model for waiting time partial backlogging when demand is constant and deterioration rate is constant. They made Abad [1996, 2001] more realistic and applicable in practice.

In this paper, an inventory model for deteriorating items is developed with time dependent demand and constant rate of deterioration. Shortages are allowed and partially backlogged; backlogging rate is variable and is dependent on the length of the next replenishment.

2. Assumption and Notations

The mathematical model is based on the following notations and assumptions.

2.1 Notations

- A the ordering cost per order.
- C the purchase cost per unit.
- θ the deterioration rate.
- h the inventory carrying cost per unit per time unit.
- π_b the backordered cost per unit short per time unit.
- π_L the cost of lost sales per unit.
- t_1 the time at which the inventory level reaches zero, $t_1 \geq 0$
- t_2 the length of period during which shortages are allowed, $t_2 \geq 0$
- T $(= t_1 + t_2)$ the length of cycle time
- IM the maximum inventory level during [0, T].
- IB the maximum inventory level during shortage period.
- Q (= IM + IB) the order quantity during a cycle of length T.
- $I_1(t)$ the level of positive inventory at time t, $0 \leq t \leq t_1$
- $I_2(t)$ the level of negative inventory at time t, $t_1 \leq t \leq t_1 + t_2$
- TC (t_1,t_2) the total cost per time unit.

2.2 Assumptions

- The demand rate is time dependent that is if ‘a’ is fix fraction of demand and ‘b’ is that fraction of demand which is vary with time then demand function is $f(t) = a + b\,t$, where $a>0$, $b>0$.
- Shortages are allowed and partially backlogged.
- The lead time is zero.
- The replenishment rate is infinite.
- The planning horizon is infinite.
- The deterioration rate is constant.
- During stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for negative inventory is, $B(t) = \dfrac{1}{1+\delta(T-t)}$ where δ is backlogging parameter and (T-t) is waiting time ($t_1 \leq t \leq T$).

3.0 Mathematical Model

The rate of change of inventory during positive stock period $[0,t_1]$ and shortage period $[t_1,T]$ is governed by the differential equations

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -(a+bt) \qquad 0 \le t \le t_1 \qquad \ldots (1)$$

$$\frac{dI_2(t)}{dt} = \frac{-(a+bt)}{1+\delta(T-t)} \qquad t_1 \le t \le T \qquad \ldots (2)$$

With boundary condition
$I_1(t) = I_2(t) = 0$ at $t=t_1$ and $I_1(t) = IM$ at $t=0$

4.0 Analytical Solution

Case I: Inventory level without shortage

During the period $[0, t_1]$, the inventory depletes due to the deterioration and demand. Hence, the inventory level at any time during $[0, t_1]$ is described by differential equation

$$\frac{dI_1(t)}{dt} + \theta_1(t) = -a \qquad 0 \le t \le t_1 \qquad \ldots (3)$$

With the boundary condition $I_1(t_1) = 0$ at $t=t_1$
The solution of equation (3) is

$$I_1(t) = -\frac{a}{\theta} - \frac{b}{\theta}\left(t - \frac{1}{\theta}\right) + e^{\theta(t_1 - t)}\left[\frac{a}{\theta} - \frac{b}{\theta}\left(t_1 - \frac{1}{\theta}\right)\right]; \qquad 0 \le t \le t_1 \qquad \ldots (4)$$

Case II: Inventory level with shortage

During the interval $[t_1, T]$ the inventory level depends on demand and a fraction of demand is backlogged. The state of inventory during $[t_1, T]$ can be represented by the differential equation

$$\frac{dI_2(t)}{dt} = \frac{-(a+bt)}{1+\delta(t_1+t_2-t)}; t_1 \le t \le t_1 + t_2 \qquad \ldots (5)$$

With the boundary condition $I_2(t_1) = 0$ at $t=t_1$
The Solution of equation (5) is

$$I_2(t) = \frac{a}{\delta}\log\frac{1+\delta(t_1+t_2-t)}{1+\delta t_2} + \frac{b[1+\delta(t_1+t_2)]}{\delta^2}\log\frac{[1+\delta(t_1+t_2-t)]}{1+\delta t_2} - \frac{b(t_1-t)}{\delta} \qquad \ldots (6)$$

Therefore the total cost per replenishment cycle consists of the following components:
Inventory holding cost per cycle;

$$IHC = h\int_{0}^{t_1} I_1(t)dt$$

$$IHC = -\frac{1}{2}\left(\frac{1}{\theta^3}\left(h(-2e^{t_1\theta}a\theta - 2e^{t_1\theta}bt_1\theta + 2e^{t_1\theta}b + 2a\theta^2 t_1 + b\theta^2 t_1^2 + 2a\theta - 2b)\right)\right) \quad \ldots (7)$$

Backordered cost per cycle;

$$\text{BC} = \pi_b\left(\int_{t_1}^{t_1+t_2} -I_2(t)dt\right)$$

$$\text{BC} = \pi_b\left(\begin{array}{l}\left(\frac{1}{2\delta^3}(2at_2\delta^2 + bt_2^2\delta^2 + 2bt_1t_2\delta^2 + 2b\delta t_2 + 2bt_2\delta\log(\frac{1}{1+\delta t_2})\right. \\ \left. +2b\log(\frac{1}{1+\delta t_2}) + 2a\delta\log(\frac{1}{1+\delta t_2}) + 2bt_1\delta\log(\frac{1}{1+\delta t_2}))\right)\end{array}\right) \quad \ldots (8)$$

Lost sales cost per cycle;

$$\text{LS} = \pi_l\left(\int_{t_1}^{t_1+t_2}\left((1 - \frac{1}{1+\delta\left(t_1+t_2-t\right)})\left(a+bt\right)\right)dt\right)$$

LS=

$$\pi_l\left[\begin{array}{l}\left(\frac{1}{2\delta^2}(2at2\delta2 + 2bt1t2\delta2 + bt_2^2\delta2 - 2a\delta\log(1+\delta t_2)\right. \\ \left. -2b\log(1+\delta t_2) - 2b\delta t1\log(1+\delta t_2) - 2b\delta t2\log(1+\delta t_2) + 2b\delta t2)\right)\end{array}\right] \quad \ldots (9)$$

Purchase cost per cycle = (purchase cost per unit) X (Order quantity in one cycle)

PC = C.Q

When t = 0 the level of inventory is maximum and it is denoted by IM (= $I_1(0)$) then from the equation (4)

$$IM = -\frac{a}{\theta} + \frac{b}{\theta^2} + e^{\theta t_1}\left[\frac{a}{\theta} - \frac{b}{\theta}(t1 - \frac{1}{\theta})\right] \quad \ldots (10)$$

The maximum backordered inventory is obtained at t = $t_1 + t_2$ then from the equation (6)

$$\text{IB} = -I_2(t_1+t_2)$$

$$IB = -[\frac{a}{\delta}\log\frac{1}{1+\delta t_2} + \frac{b[1+\delta(t_1+t_2)]}{\delta^2}\log\frac{1}{1+\delta t_2} + \frac{bt_2}{\delta}] \quad \ldots (11)$$

Thus the order size during total time interval [0,T]

Q = IM + IB

Now from equations (10) and (11)

$$Q = \begin{bmatrix} -\frac{a}{\theta}+\frac{b}{\theta^2}+e^{\theta t_1}\left[\frac{a}{\theta}-\frac{b}{\theta}(t1-\frac{1}{\theta})\right]-\frac{a}{\delta}\log\frac{1}{1+\delta t_2} \\ -\frac{b[1+\delta(t_1+t_2)]}{\delta^2}\log(1+\delta t_2)-\frac{bt2}{\delta} \end{bmatrix}$$

Thus PC = C.Q

$$= C\begin{bmatrix} -\frac{a}{\theta}+\frac{b}{\theta^2}+e^{\theta t_1}\left[\frac{a}{\theta}-\frac{b}{\theta}(t1-\frac{1}{\theta})\right]-\frac{a}{\delta}\log\frac{1}{1+\delta t_2} \\ -\frac{b[1+\delta(t_1+t_2)]}{\delta^2}\log(1+\delta t_2)-\frac{bt2}{\delta} \end{bmatrix} \quad \text{... (12)}$$

Therefore the total cost per time unit is given by,

$= \frac{1}{(t_1+t_2)}$ [Ordering cost + carrying cost + backordering cost + lost sale cost + purchase Cost]

$$TC\left(t_{1,} t\right) = \frac{1}{(t_1+t_2)} \text{ [OC+ IHC + BC + LS + PC]}$$

Now putting the values in this equation of OC, IHC, BC, LS and PC then,

$$TC\left(t_{1,} t\right) = \frac{1}{(t_1+t_2)}\left(\begin{array}{l} A-\frac{1}{2\theta^3}(h(-2e^{t_1\theta}a\theta--2e^{t_1\theta}bt_1\theta+-2e^{t_1\theta}b+2a\theta^2t_1+b\theta^2t_1^2+2a\theta-2b)) \\ +\frac{1}{2\delta^2}(\pi_l(2at_2\delta^2+2bt_1t_2\delta^2+bt_2\delta^2-2a\delta\log(1+\delta t_2)-2b\log(1+\delta t_2) \\ -2bt_1\delta\log(1+\delta t_2)-2bt_2\delta\log(1+\delta t_2)+2b\delta t_2 \\ ))+\frac{1}{2\delta^3}(\pi_b(2at_2\delta^2+bt_2^2\delta^2+2bt_1t_2\delta^2+2b\delta t_2+ \\ 2bt_2\log(\frac{1}{1+\delta t_2})\delta+2b\log(\frac{1}{1+\delta t_2})+2a\log(\frac{1}{1+\delta t_2})\delta+2b\log(\frac{1}{1+\delta t_2})t_1\delta)) \\ +c\left(\begin{bmatrix} -\frac{a}{\theta}+\frac{b}{\theta^2}+e^{\theta t_1}\left[\frac{a}{\theta}-\frac{b}{\theta}(t1-\frac{1}{\theta})\right]-\frac{a}{\delta}\log\frac{1}{1+\delta t_2} \\ -\frac{b[1+\delta(t_1+t_2)]}{\delta^2}\log(1+\delta t_2)-\frac{bt2}{\delta} \end{bmatrix}\right) \end{array}\right) \quad \text{… (13)}$$

The necessary condition for the total cost per time unit, to be minimize is

$$\frac{\partial TC}{\partial t_1} = 0 \text{ and } \frac{\partial TC}{\partial t_2} = 0$$

Provided

$$\left(\frac{\partial^2 TC}{\partial t_1^2}\right)\left(\frac{\partial^2 TC}{\partial t_2^2}\right) - \left(\frac{\partial^2 TC}{\partial t_1 \partial t_2}\right) > 0 \qquad \ldots (14)$$

5.0 Sensitivity Analysis

Consider an inventory system with the following parameter in proper unit A=2500, h=.5, C=4, pib=12, pil=15, delta=8, a=25, b=20.0, theta=.005 .The computer output by using maple mathematical software is t1=5.4 t2=0.4 and TC= 915.30. i.e. the value of t1 at which the inventory level become zero is 5.40 unit and shortage period is 0.04 unit .The variation in the parameter is as follows

Table 5.1 Variation in parameter δ

δ	t_1	t_2	TC
6.4	5.40	.04	915.07
8.0	5.40	.04	915.30
8.8	5.40	.03	915.39
9.2	5.41	.03	915.44

Table-5.2 Variation in parameter θ

θ	t_1	t_2	TC
.0040	5.42	.04	913.99
.0045	5.41	.04	914.65
.0050	5.40	.04	915.30
.0055	5.40	.04	915.96

Table- 5.3 Variation in parameter b

b	t_1	t_2	TC
21.0	5.30	.04	931.15
20.0	5.40	.04	915.30
18.0	5.62	.04	882.44
16.0	5.87	.04	847.71

Table- 5.4 Variation in parameter a

a	t_1	t_2	TC
26.0	5.40	.04	888.22
25.0	5.40	.04	888.21
22.5	5.42	.04	888.20
20.0	5.44	.04	888.18

From table 5.1, 5.2, 5.3 and 5.4 we observed that the total cost increases if we increases the parameter a,b , θ and δ .it's also observed that the parameter a and b is more sensitive than the parameter θ and δ

If we plot the total cost function (13) with some values of t_1 and t_2 s.t., t_1= 5.0 to 5.80 with equal interval t2 = 0.01 to.09.Then we get a three dimensional convex

Graph of TC given by the figure (5.1)

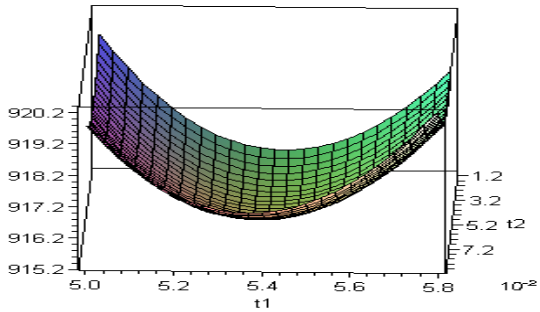

Fig-5.1

6.0 Concluding Remarks

In this paper, we developed a model for deteriorating item with time dependent demand and partial backlogging and give analytical solution of the model that minimize the total inventory cost. The deterioration factor taken into consideration in the present model, as almost all items undergo either direct spoilage (like fruits, vegetable etc) or physical decay (in case of radioactive substance etc.) in the course time, deterioration is natural feature in the inventory system. The model is very useful in the situation in which the demand rate is depending upon the time.

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