Applied Mathematical Sciences, Vol. 6, 2012, no. 47, 2345 - 2355

Ones Assignment Method for Solving Assignment Problems

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Abstract

Assignment problem is an important subject discussed in real physical world. We endeavor in this paper to introduce a new approach to assignment problem namely, ones assignment method, for solving a wide rang of such problems.
This method offers significant advantages over similar methods, in the process, first we define the assignment matrix, then by using determinant representation we obtain a reduced matrix which has at least one 1 in each row and columns. Then by using the new method, we obtain an optimal solution for assignment problem by assigning ones to each row and each column. The new method is based on creating some ones in the assignment matrix and then try to find a complete assignment to there ones.
The proposed method is a systematic procedure, easy to apply and can be utilized for all types of assignment problem with maximize or minimize objective functions.
At the end, this method is illustrated with some numerical examples.

Mathematics Subject Classification: 90C08, 90C10

Keywords: Assignment problem, Linear integer programming

1 Introduction

An important topic, put forward immediately after the transportation problem, is the assignment problem. This is particularly important in the theory of decision making. The assignment problem is one of the earliest application of linear integer programming problem. Different methods have been presented for assignment problem and various articles have been published on the subject. See [1], [2] and [3] for the history of these methods.

A considerable number of methods has been so far presented for assignment problem in which the Hungarian method is more convenient method among them. This iterative method is based on add or subtract a constant to every element of a row or column of the cost matrix, in a minimization model and create some zeros in the given cost matrix and then try to find a complete assignment in terms of zeros. By a complete assignment for a cost matrix $n \times n$, we mean an assignment plan containing exactly n assigned independent zeros, one in each row and one in each column. The main concept of assignment problem is to find the optimum allocation of a number of resources to an equal number of demand points. An assignment plan is optimal if optimizes the total cost or effectiveness of doing all the jobs.

This paper attempts to propose a method for solving assignment problem which is different from the preceding methods.

2 Mathematical formulation of assignment problem

Mathematically an assignment problem can be stated as follows:

Optimize

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1}$$

subject to

$$\left.\begin{array}{lll} \sum_{j=1}^{n} x_{ij} = 1, & i = 1,\cdots,n & \\ \sum_{i=1}^{n} x_{ij} = 1, & j = 1,\cdots,n & \\ x_{ij} = 0 \quad or \quad 1, & i = 1,\cdots,n \quad , & j = 1,\cdots,n. \end{array}\right\} \tag{2}$$

where c_{ij} is the cost or effectiveness of assigning ith job to jth machine, and x_{ij} is to be some positive integer or zero, and the only possible integer is one, so the condition of $x_{ij} = 0$ or 1, is automatically satisfied.
Associated to each assignment problem there is a matrix called cost or effectiveness matrix $[c_{ij}]$ where c_{ij} is the cost of assigning ith job to jth facility. In this paper we call it assignment matrix, and represent it as follows:

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ n \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & n \end{array} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{pmatrix} \end{array}$$

which is always a square matrix, thus each task can be assigned to only one machine.
In fact any solution of this assignment problem will contain exactly m non-zero positive individual allocations.
A customary and convenient method , termed as "assignment algorithm" has been developed for such problems. This iterative method is known as Hungarian assignment method. It is based on add or subtract a constant to every element of a row or column of the cost matrix in a minimization model , and create some zeros in the given cost matrix and then try to find a complete assignment in terms of zeros. In fact our aim is to create ones in place of zeroes, and try to assign them in our problem.

3 A new approach for solving assignment problem

This section presents a new method to solve the assignment problem which is different from the preceding method. We call it "one- assignment method", because of making assignment in terms of ones.
The new method is based on creating some ones in the assignment matrix and then try to find a complete assignment in terms of ones. By a complete assignment we mean an assignment plan containing exactly m assigned independent ones, one in each row and one in each column.
Now, consider the assignment matrix where c_{ij} is the cost or effectiveness of assigning ith job to jth machine.

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ n \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & n \end{array} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{pmatrix} \end{array}$$

The new algorithm is as follows:

let (1-2) be an assignment problem in which the objective function can be minimized or maximized.

step 1.
In a minimization (maximization) case, find the minimum (maximum) element of each row in the assignment matrix (say a_i) and write it on the right hand side of the matrix.

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & \dots & n \end{matrix} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{pmatrix} \begin{matrix} a_1 \\ a_2 \\ \\ a_n \end{matrix} \end{array}$$

Then divide each element of ith row of the matrix by a_i. These operations create at least one ones in each rows.

In term of ones for each row and column do assignment, otherwise go to step 2.
step 2.
Find the minimum(maximum) element of each column in assignment matrix (say b_j), and write it below jth column. Then divide each element of jth column of the matrix by b_j.
These operations create at least one ones in each columns. Make assignment in terms of ones. If no feasible assignment can be achieved from step (1) and (2) then go to step 3.

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & \dots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \\ n \end{matrix} & \begin{pmatrix} c_{11}/a_1 & c_{12}/a_1 & c_{13}/a_1 & \dots & c_{1n}/a_1 \\ c_{21}/a_2 & c_{22}/a_2 & c_{23}/a_2 & \dots & c_{2n}/a_2 \\ \vdots & \vdots & \ddots & \vdots & \\ c_{n1}/a_n & c_{n2}/a_n & c_{n3}/a_n & \dots & c_{nn}/a_n \end{pmatrix} \\ & \begin{matrix} b_1 & b_2 & b_3 & \dots & b_n \end{matrix} \end{array}$$

Note: In a maximization case, the end of step 2 we have a fuzzy matrix, which all elements are belong to $[0, 1]$, and the greatest element is one [4].
step 3.
Draw the minimum number of lines to cover all the ones of the matrix. If the number of drawn lines less than n, then the complete assignment is not possible, while if the number of lines is exactly equal to n, then the complete assignment is obtained.

step 4.
If a complete assignment program is not possible in step 3 , then select the smallest (largest) element (say d_{ij}) out of those which do not lie on any of the lines in the above matrix. Then divide by d_{ij} each element of the uncovered rows or columns, which d_{ij} lies on it. This operation create some new ones to this row or column.
If still a complete optimal assignment is not achieved in this new matrix, then use step 4 and 3 iteratively. By repeating the same procedure the optimal assignment will be obtained.

Priority, plays an important role in this method, When we want to assign the ones.

Priority rule.
For maximization (minimization) assignment problem, assign the ones on the rows which have greatest (smallest) element on the right hand side, respectively.

One question arise here, what to do with non square matrix? To make square, a non square matrix, we add one artificial row or column which all elements are one. Thus we solve the problem with the new matrix, by using the new method. The matrix after performing the steps reduces to a matrix which has ones in each rows and columns. So , the optimal assignment has been reached.

4 Mathematical concept of the subject

One of the operations associated with matrices is calculation of scalar value known as the determinant of a square matrix [6]. Here we do not want calculate the determinant of a matrix, only we want to use the properties of the determinant operator, when we use customary and common notation, for the determinant of the matrix.
Here we use the notation $|A|$, for representation of the determinant of matrix A. Several basic properties of determinant are useful but one of them useful for this method. This is "factorization".
Proposition 1:
When λ (a non zero scalar) is a factor of a row (column) of $|A|$, then it is also a factor of $|A|$. That is

$$|A| = \lambda.|A_\lambda|$$

which A_λ is the matrix A with λ factored out of a row or a column of A.
If A is an $n \times n$ matrix and $\lambda_i, i = 1, 2, ..., n$ is a factor of ith row of $|A|$, then $\lambda = \lambda_1.\lambda_2....\lambda_n$ is also a factor of $|A|$. That is

$$|A| = \lambda.|A_\lambda| = \lambda_1.\lambda_2....\lambda_n.|A_{\lambda_1.\lambda_2....\lambda_n}|$$

Similarly, if A is an $n \times n$ matrix and $\mu_j, j = 1, 2, ..., n$ is a factor of jth column of $|A|$, then $\mu = \mu_1.\mu_2....\mu_n$ is also a factor of $|A|$.

$$|A| = \mu.|A_\mu| = \mu_1.\mu_2....\mu_n|A_{\mu_1.\mu_2....\mu_n}|$$

which $A_{\mu_1.\mu_2....\mu_n}$ is the matrix A with μ_j factored out of jth column, j= 1,2,...,n.
Proposition 2:
If A is an $n \times n$ matrix and λ is a factor of rows of $|A|$, and μ is a factor of columns of $|A|$, then $\lambda.\mu$ is also a factor of $|A|$. That is ,

$$|A| = \lambda.\mu|A_{\lambda.\mu}|$$

which $A_{\lambda\mu}$ is the matrix A with λ factored out of a row and μ factored out of a column of A .

Note : In fact, when we apply the ones assignment method to solve an assignment problem, we use the above propositions to reduce the matrix of the problem in to a matrix which has enough ones in each rows and each columns to assign them. Here we emphasis that the locations of the ones are important to assign, and when we want to calculate the objective function of the problem, we use the real values of the assigned elements in initial matrix.

5 Numerical examples

The following examples may be helpful to clarify the proposed method:

Example 1: Consider the following assignment problem. Assign the five jobs to the three machines so as to minimize the total cost.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 12 & 8 & 7 & 15 & 4 \\ 2 & 7 & 9 & 1 & 14 & 10 \\ 3 & 9 & 6 & 12 & 6 & 7 \\ 4 & 7 & 6 & 14 & 6 & 10 \\ 5 & 9 & 6 & 12 & 10 & 6 \end{array}$$

Find the minimum element of each row in the assignment matrix (say a_i) and write it on the right hand side of the matrix, as follows:

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & min \\ \hline 1 & 12 & 8 & 7 & 15 & 4 & 4 \\ 2 & 7 & 9 & 1 & 14 & 10 & 1 \\ 3 & 9 & 6 & 12 & 6 & 7 & 6 \\ 4 & 7 & 6 & 14 & 6 & 10 & 6 \\ 5 & 9 & 6 & 12 & 10 & 6 & 6 \end{array}$$

Then divide each element of ith row of the matrix by a_i. These operations create ones to each rows, and the matrix reduces to following matrix.

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & min \\ \hline 1 & 3 & 2 & 7/4 & 15/4 & 1 & 4 \\ 2 & 7 & 9 & 1 & 14 & 10 & 1 \\ 3 & 3/2 & 1 & 2 & 1 & 7/6 & 6 \\ 4 & 7/6 & 1 & 7/3 & 1 & 5/3 & 6 \\ 5 & 3/2 & 1 & 2 & 5/3 & 1 & 6 \end{array}$$

Now find the minimum element of each column in assignment matrix (say b_j), and write it below that column. Then divide each element of jth column of the matrix by b_j.

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & min \\ \hline 1 & 3 & 2 & 7/4 & 15/4 & 1 & 4 \\ 2 & 7 & 9 & 1 & 14 & 10 & 1 \\ 3 & 3/2 & 1 & 2 & 1 & 7/6 & 6 \\ 4 & 7/6 & 1 & 7/3 & 1 & 5/3 & 6 \\ 5 & 3/2 & 1 & 2 & 5/3 & 1 & 6 \\ \hline min & 7/6 & 1 & 1 & 1 & 1 & \end{array}$$

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & min \\ \hline 1 & 18/7 & 2 & 7/4 & 15/4 & 1 & 4 \\ 2 & 6 & 9 & 1 & 14 & 10 & 1 \\ 3 & 18/14 & 1 & 2 & 1 & 7/6 & 6 \\ 4 & 1 & 1 & 7/3 & 1 & 5/3 & 6 \\ 5 & 18/14 & 1 & 2 & 5/3 & 1 & 6 \\ \hline min & 7/6 & 1 & 1 & 1 & 1 & \end{array}$$

The minimum number of lines required to pass through all the ones of the matrix is 5.

$$\begin{pmatrix} 18/7 & 2 & 7/4 & 15/4 & 1 \\ \sout{6} & \sout{9} & \sout{1} & \sout{14} & \sout{10} \\ \sout{18/14} & \sout{1} & \sout{2} & \sout{1} & \sout{7/6} \\ \sout{1} & \sout{1} & \sout{7/3} & \sout{1} & \sout{5/3} \\ \sout{18/14} & \sout{1} & \sout{2} & \sout{5/3} & \sout{1} \end{pmatrix}$$

So, the complete assignment is possible,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 18/7 & 2 & 7/4 & 15/4 & \boxed{1} \\ 6 & 9 & \boxed{1} & 14 & 10 \\ 18/14 & 1 & 2 & \boxed{1} & 7/6 \\ \boxed{1} & 1 & 7/3 & 1 & 5/3 \\ 18/14 & \boxed{1} & 2 & 5/3 & 1 \end{pmatrix} \end{array}$$

and we can assign the ones and the solution is $(1,5),(2,3),(3,4),(4,1),(5,2)$ and minimum cost is 24.

Example 2: A company has 5 jobs to be done. The following matrix shows the return of assigning ith machine to the jth job. Assign the five jobs to the five machines so as to maximize the total return.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 5 & 11 & 10 & 12 & 4 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 12 & 5 & 14 & 6 \\ 6 & 14 & 4 & 11 & 7 \\ 7 & 9 & 8 & 12 & 5 \end{pmatrix} \end{array}$$

step 1. Find the maximum element of each row in the assignment matrix (say a_i) and write it on the right hand side of the matrix, as follows:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 5 & 11 & 10 & 12 & 4 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 12 & 5 & 14 & 6 \\ 6 & 14 & 4 & 11 & 7 \\ 7 & 9 & 8 & 12 & 5 \end{pmatrix} \end{array} \begin{array}{c} max \\ 12 \\ 6 \\ 14 \\ 14 \\ 12 \end{array}$$

Then divide each element of ith row of the matrix by a_i. These operations create ones to each rows, and the matrix reduces to following matrix.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.42 & 0.92 & 0.83 & 1 & 0.33 \\ 0.33 & 0.66 & 1 & 0.5 & 0.83 \\ 0.21 & 0.86 & 0.36 & 1 & 0.43 \\ 0.43 & 1 & 0.28 & 0.78 & 0.5 \\ 0.58 & 0.75 & 0.66 & 1 & 0.46 \end{pmatrix} \end{array} \begin{array}{c} max \\ 12 \\ 6 \\ 14 \\ 14 \\ 12 \end{array}$$

step 2. Since we can not make assignment in terms of the ones, now find the maximum element of each column in assignment matrix (say b_j), and write it below that column. Then divide each element of jth column of the matrix by b_j.

These operations create ones to each columns, and matrix reduces to following matrix , feasible assignment can not be secured from step (1) and (2) so, go to step 3.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\ max \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.42 & 0.92 & 0.83 & 1 & 0.33 \\ 0.33 & 0.66 & 1 & 0.5 & 0.83 \\ 0.21 & 0.86 & 0.36 & 1 & 0.43 \\ 0.43 & 1 & 0.28 & 0.78 & 0.5 \\ 0.58 & 0.75 & 0.66 & 1 & 0.46 \end{pmatrix} \\ \begin{array}{ccccc} 0.58 & 0.92 & 1 & 1 & 0.83 \end{array} \end{array} \begin{array}{c} max \\ 12 \\ 6 \\ 14 \\ 14 \\ 12 \\ \\ \end{array}$$

step 3. Draw the minimum number of lines required to pass through all the ones of the matrix. Clearly the four lines are passing through all the ones. Since the number of lines is not equal to the order of the matrix, the optimal assignment has not been reached.

$$\begin{pmatrix} 0.71 & 0.92 & 0.83 & 1 & 0.4 \\ 0.57 & 0.66 & 1 & 0.5 & 1 \\ 0.32 & 0.86 & 0.36 & 1 & 0.51 \\ 0.73 & 1 & 0.28 & 0.78 & 0.6 \\ 1 & 0.75 & 0.66 & 1 & 0.5 \end{pmatrix}$$

We perform the step 4.

step 4. Select the largest element (say d_{ij}) out of those which do not lie on any of the lines in the above matrix. It is in the third column $d_{ij} = 0.83$. Then divide each element of the third column, which d_{ij} is on it. This operation creates a new one to this column.

$$\begin{pmatrix} 0.71 & 0.92 & 1 & 1 & 0.4 \\ 0.57 & 0.66 & 1.2 & 0.5 & 1 \\ 0.32 & 0.86 & 0.43 & 1 & 0.51 \\ 0.73 & 1 & 0.34 & 0.78 & 0.6 \\ 1 & 0.75 & 0.79 & 1 & 0.5 \end{pmatrix}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.71 & 0.92 & \boxed{1} & 1 & 0.4 \\ 0.57 & 0.66 & 1.2 & 0.5 & \boxed{1} \\ 0.32 & 0.86 & 0.43 & \boxed{1} & 0.51 \\ 0.73 & \boxed{1} & 0.34 & 0.78 & 0.6 \\ \boxed{1} & 0.75 & 0.79 & 1 & 0.5 \end{pmatrix} \end{array} \begin{array}{c} max \\ 12 \\ 6 \\ 14 \\ 14 \\ 12 \end{array}$$

Now, we can assign the ones, it is based on priority rule. Priority rule is assigning one on the rows which have greatest element on the right hand side, respectively.

The details of this program are as follows:

Machine 3 assigns to job 4 profit 14
Machine 4 assigns to job 2 profit 14
Machine 5 assigns to job 1 profit 7
Machine 1 assigns to job 3 profit 10
Machine 2 assigns to job 5 profit 5

so the optimal assignment has been reached, and total profit according to this plan is 10+5+14+14+7=50.

6 Conclusion

In this paper, a new and simple method was introduced for solving assignment problem. This method can be used for all kinds of assignment problems, whether maximize or minimize objective function. The new method is based on creating some ones in the assignment matrix, and find an assignment in terms of the ones.

References

[1] M. S. Bazarra, John J. Jarvis, Hanif D. Sherali, *Linear programming and network flows*, (2005).

[2] B. s. Goel, S. K. Mittal, *Operations Research*, Fifth Ed., (1982) 2405-2416.

[3] Hamdy A. Taha, *Operations Research, an introduction*, 8th Ed. (2007).

[4] H. J. Zimmermann, *Fuzzy set theory and its Applications*, third Ed., kluwer Academic, Boston, 1996.

[5] Anshuman Sahu, Rudrajit Tapador, *Solving the assignment problem using genetic algorithm and simulated annealing*, IJAM, (2007).

[6] Shayle R. Searle, *Matrix algebra useful for statistics*, John Wiley, (2006).

Received: November, 2011