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Improving the Accuracy of the Adomian Decomposition Method for Solving Nonlinear Equations

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Abstract

In this paper, we apply the Shanks transformation on the Adomian decomposition method (ADM) to solve nonlinear equations. Some numerical illustrations are given to show the effectiveness of the proposed modification. The numerical results represent that utilization of this method in identical conditions gives more suitable solutions for nonlinear equations in comparison with those obtained from the ADM.

Keywords: Adomian decomposition method; Shanks transformation; nonlinear equations

1 Introduction

One of the most basic problems in numerical analysis is that of finding the solution of the equation

$$f(x) = 0 \tag{1}$$

for a given function f which is sufficiently smooth in the neighborhood of a simple root α. In most cases, it is difficult to obtain an analytical solution of (1). Therefore, the exploitation of numerical techniques for solving such equations becomes a main subject of considerable interests. Probably the most well-known and widely used algorithm to find a root α is Newton's method [1, 2]. In recent years, there have been some developments in the study of Newton-like iterative methods. To obtain these iterative methods, the ADM [12] as well as the other more general methods such as the homotopy perturbation method [13] and the homotopy analysis method [14] plays an important role in the process of numerical approximation. The ADM considers the approximate solution of a nonlinear equation as an infinite series converging to the accurate solution. Over the past few years, the ADM has been applied to solve a wide range of problems, both deterministic and stochastic, linear and nonlinear, arising from physics, chemistry, biology, engineering etc [15, 16].

Recently several authors have proposed a variety of modifications on the ADM for solving $f(x) = 0$. E. Babolian et al in [4] obtained the solution of nonlinear equations by modified Adomian decomposition method. In another work, he applied the restarted Adomian decomposition method to solve the algebraic equation [5]. S. Abbasbandy in [6] applied improved Newton-Raphson method for nonlinear equations by modified Adomian decomposition method.

In this paper, we apply the Shanks transformation on the ADM to improve the accuracy of the approximate solutions to solve nonlinear equations.

In the next section, we introduce the ADM for nonlinear equations. The Shanks transformation is introduced in Section 3. In Section 4, we investigate several numerical examples which demonstrate the effectiveness of the proposed method in this paper. In Section 5, we summarize our findings.

2 ADM for nonlinear equations

Consider the nonlinear equation (1), which can be written in the following canonical form

$$x = c + N(x), \qquad x \epsilon R, \tag{2}$$

where $N(x)$ is a nonlinear function and c is a constant. The ADM consists of calculating the solution in the series form

$$x = \sum_{n=0}^{\infty} x_n, \tag{3}$$

where the components x_n are to be calculated recursively and nonlinear function is decomposed as

$$N(x) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

where the A_n's are functions called the Adomian's polynomials depending on $x_0, x_1, \ldots, x_n$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i x_i)]_{\lambda=0}, \quad n = 0, 1, \ldots \tag{5}$$

The polynomials A_n's are generated for all kinds of nonlinearity [17]. The first few polynomials are given

$$A_0 = N(x_0) \tag{6}$$

$$A_1 = x_1 N'(x_0) \tag{7}$$

$$A_2 = x_2 N'(x_0) + \frac{1}{2} x_1^2 N''(x_0). \tag{8}$$

By substituting Eqs. (3) and (4) into Eq. (2), we have

$$\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n \tag{9}$$

Each term of series (9) is given by the recurrence relation

$$x_0 = c, \tag{10}$$

$$x_{n+1} = A_n. \tag{11}$$

In practice, not all terms of series (3) can be determined, and hence the solution will be approximated by the truncated series

$$\Omega_k = \sum_{n=0}^{k-1} x_n. \tag{12}$$

In computing x using any mathematical software, as n increases the number of terms in the expression for A_n increases and this causes propagation of round off errors. On the other hand, the factor $1/n!$ in the formula of A_n makes it very small, so that its contribution to x is negligible. Hence, the first few terms of the series $\sum_{n=0}^{\infty} x_n$ determine the accuracy of the approximate solution. onsidering this, we apply the Shanks transformation on the ADM to improve the accuracy of the approximate solutions dramatically.

3 Shanks transformation on the ADM

In numerical analysis, the Shanks transform [11] is a nonlinear transform which can be very effective, particularly in accelerating theconvergence of slowly converging series. It has even been applied to diverging series which seems contradictory. The Shanks transformation $T(x_n)$ of the terms of the series (3) is defined as

$$T(x_n) = \frac{x_{n+1} x_{n-1} - x_n^2}{x_{n+1} + x_{n-1} - 2x_n}. \tag{13}$$

The sequence $T(x_n)$ often converges more rapidly than the sequence x_n to the exact solution.

4 Numerical applications

Example 3.1 Consider the nonlinear equation

$$x^2 - (1-x)^5 = 0, \tag{14}$$

which has the exact solution $\alpha = 0.345955(6D)$. To apply the ADM, we rewrite Eq. (14) in the cononical form

$$x = 0.2 + 1.8x^2 - 2x^3 + x^4 - 0.2x^5. \tag{15}$$

By substituting Eqs.(3) and (4) into Eq. (15), we obtain

$$\sum_{n=0}^{\infty} x_n = 0.2 + 1.8\sum_{n=0}^{\infty} A_n - 2\sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n - 0.2\sum_{n=0}^{\infty} D_n, \tag{16}$$

where A_n, B_n, C_n, and D_n's are Adomian's polynomials of order $2, 3, 4$, and 5. Based on the structure of the ADM, each term of series (16) is given by the recurrence relation

$$x_0 = 0.2, \tag{17}$$

$$x_{n+1} = 1.8A_n - 2B_n + C_n - 0.2D_n, \quad n \geq 0. \tag{18}$$

We approximate x by $\Omega_k = \sum_{n=0}^{k-1} x_n$, and hence

$$\Omega_1 = x_0 = 0.2, \tag{19}$$

$$\Omega_2 = x_0 + x_1 = 0.257536, \tag{20}$$

$$\Omega_3 = x_0 + x_1 + x_2 = 0.286902, \tag{21}$$

$$\vdots$$

$$\Omega_{10} = x_0 + x_1 + ... + x_9 = 0.338972, \tag{22}$$

which is an approximation of the exact solution of Eq. (14). Obviously, the series $\Omega_n, n = 1, \ldots, 10$ has a slow convergence rate to the exact solution. The Shanks transform is an efficient relation that can accelerate the convergence rate of the series. Applying the Shanks transform (13) to the sequence $\{\Omega_k\}$ for $k = 1, 2, \ldots, 10$, we obtain the data given in Table 1.

As can be seen in the last column of Table 1, the obtained solution equals to 0.345955 up to 5 digits.

Example 3.2 Consider the nonlinear equation

$$x^3 - 7x^2 + 14x - 6 = 0, \tag{23}$$

with the exact solution $\alpha = 0.585786(6D)$. An equivalent expression for Eq. (23) is

$$x = \frac{3}{7} + \frac{1}{2}x^2 - \frac{1}{14}x^3. \tag{24}$$

n	Ω_n	A_n	B_n	C_n	D_n
1	0.200000				
2	0.257536	0.317516			
3	0.286902	0.331560	0.343721		
4	0.304619	0.338078	0.344964	0.345857	
5	0.316202	0.341426	0.345484	0.345924	0.345961
6	0.324140	0.343260	0.345722	0.345948	0.345961
7	0.329749	0.344312	0.345838	0.345956	
8	0.333799	0.344934	0.345896		
9	0.336768	0.345312			
10	0.338972				

Table 1: *The results of applying Shanks transform on* $\Omega_n, n = 1, \ldots, 10$.

By substituting Eqs. (3) and (4) into Eq. (24), we obtain

$$\sum_{n=0}^{\infty} x_n = \frac{3}{7} + \frac{1}{2}\sum_{n=0}^{\infty} A_n - \frac{1}{14}\sum_{n=0}^{\infty} B_n \tag{25}$$

where A_n and B_n are Adomian's polynomials of order 2 and 3. Each term of series (25) is given by the recurrence relation

$$x_0 = \frac{3}{7}, \tag{26}$$

$$x_{n+1} = \frac{1}{2}A_n - \frac{1}{14}B_n, \quad n \geq 0. \tag{27}$$

We approximate x by $\Omega_k = \sum_{n=0}^{k-1} x_n$, and hence

$$\Omega_1 = x_0 = 0.428571, \tag{28}$$

$$\Omega_2 = x_0 + x_1 = 0.514786, \tag{29}$$

$$\Omega_3 = x_0 + x_1 + x_2 = 0.548341, \tag{30}$$

$$\vdots$$

$$\Omega_{10} = x_0 + x_1 + ... + x_9 = 0.584392, \tag{31}$$

which is an approximation of the exact solution Eq. (23). Applying the Shanks transform (13) to the sequence $\{\Omega_k\}$ for $k = 1, 2, \ldots, 10$, we obtain the data given in Table 2. As can be seen in the last column of Table 2, the obtained solution equals to the exact solution.

Example 3.3 Consider the nonlinear equation

$$e^{-x} - x + 2 = 0, \tag{32}$$

n	Ω_n	A_n	B_n	C_n	D_n
1	0.428571				
2	0.514786	0.569724			
3	0.548341	0.579269	0.584903		
4	0.564435	0.582812	0.585444	0.585751	
5	0.573015	0.584322	0.585640	0.585773	0.585785
6	0.577893	0.585026	0.585719	0.585781	0.585786
7	0.580790	0.585375	0.585754	0.585784	
8	0.582566	0.585557	0.585770		
9	0.583680	0.585655			
10	0.584392				

Table 2: *The results of applying Shanks transform on* $\Omega_n, n = 1, \ldots, 10$.

The exact solution of Eq. (32) is $\alpha = 2.1200282389876(13D)$. Eq. (32) in canonical form is equivalent to

$$x = 2 + e^{-x}. \tag{33}$$

By substituting Eqs. (3) and (4) into Eq. (33), we obtain

$$\sum_{n=0}^{\infty} x_n = 2 + \sum_{n=0}^{\infty} A_n. \tag{34}$$

Each term of series (34) is given by the recurrence relation

$$x_0 = 2, \tag{35}$$

$$x_{n+1} = A_n. \quad n \geq 0. \tag{36}$$

We approximate x by $\Omega_k = \sum_{n=0}^{k-1} x_n$ which gives

$$\Omega_1 = x_0 = 2, \tag{37}$$

$$\Omega_2 = x_0 + x_1 = 2.1353352832366, \tag{38}$$

$$\Omega_3 = x_0 + x_1 + x_2 = 2.1170196443479, \tag{39}$$

$$\vdots$$

$$\Omega_{10} = x_0 + x_1 + ... + x_9 = 2.1200286699020, \tag{40}$$

which is an approximation of the exact solution Eq. (32). Applying the Shanks transform (13) to the sequence $\{\Omega_k\}$ for $k = 1, 2, \cdots, 10$, we obtain the data given in Table 3.

n	Ω_n	A_n	B_n	C_n	D_n
1	2.0000000000000				
2	2.13533528323366	2.1192029220221			
3	2.11701964434790	2.1201103502707	2.1200254561784		
4	2.12073777261290	2.1200166942238	2.1200285195622	2.1200282331202	
5	2.11984320560510	2.1200302284394	2.1200282035737	2.1200282396192	2.1200282389797
6	2.12007966357270	2.1200278473346	2.1200282442604	2.1200282389099	2.1200282389886
7	2.12001330607920	2.1200283236636	2.12002822380997	2.1200282389985	
8	2.1200327165003	2.12002821936420	2.1200282391520		
9	2.1200268632422	2.1200282437852			
10	2.1200286699020				

Table 3: *The results of applying Shanks transform on* $\Omega_n, \quad n = 1, \ldots, 10$

As can be seen in the last column of Table 3, the obtained solution equals to 2.1200282389876 up to 12 digits.

Example 3.4 Consider the nonlinear equation

$$e^x - x^2 - 3x + 2 = 0, \tag{41}$$

which has the exact solution $\alpha = 0.25753(5D)$. By rewriting (41) in the canonical form, we have

$$x = \frac{2}{3} - \frac{1}{3}e^x + \frac{1}{3}x^2 \tag{42}$$

Replacing Eqs. (3) and (4) into Eq. (42), we have

$$\sum_{n=0}^{\infty} x_n = \frac{2}{3} - \frac{1}{3}\sum_{n=0}^{\infty} A_n + \frac{1}{3}\sum_{n=0}^{\infty} B_n. \tag{43}$$

Each term of series (43) is given by the recurrence relation

$$x_0 = \frac{2}{3}, \tag{44}$$

$$x_{n+1} = -\frac{1}{3}A_n + \frac{1}{3}B_n, \quad n \geq 0 \tag{45}$$

Approximating x by $\Omega_k = \sum_{n=0}^{k-1} x_n$ gives

$$\Omega_1 = x_0 = 0.666667, \tag{46}$$

$$\Omega_2 = x_0 + x_1 = 0.165570, \tag{47}$$

$$\Omega_3 = x_0 + x_1 + x_2 = 0.268195, \tag{48}$$

$$\vdots$$

$$\Omega_{10} = x_0 + x_1 + ... + x_9 = 0.256616, \tag{49}$$

n	Ω_n	A_n	B_n	C_n	D_n
1	0.666667				
2	0.165570	0.250750			
3	0.268195	0.252284	0.250211		
4	0.249365	0.258180	0.258620	0.258301	
5	0.265940	0.258589	0.258288	0.258798	0.258432
6	0.252731	0.257450	0.257342	0.257412	0.257221
7	0.260074	0.257351	0.257418	0.257244	
8	0.255747	0.257556	0.257554		
9	0.258856	0.257554			
10	0.256616				

Table 4: *The results of applying Shanks transform on* $\Omega_n, n = 1, \ldots, 10$

Applying the Shanks transform (13) to the sequence $\{\Omega_k\}$ for $k = 1, 2, \ldots, 10$, we obtain the data given in Table 4.

As can be seen in the last column of Table 4, the obtained solution equals to 0.25753 up to 4 digits.

Example 3.5 Consider the nonlinear equation

$$x^3 + 4x^2 + 8x + 8 = 0, \tag{50}$$

which has the exact solution $\alpha = -2$. By rewriting (50) in the canonical form, we obtain

$$x = -1 - \frac{1}{2}x^2 - \frac{1}{8}x^3 \tag{51}$$

Replacing Eqs. (3) and (4) into Eq. (51), we have

$$\sum_{n=0}^{\infty} x_n = -1 - \frac{1}{2}\sum_{n=0}^{\infty} A_n - \frac{1}{8}\sum_{n=0}^{\infty} B_n. \tag{52}$$

Each term of series (52) is given by the recurrence relation

$$x_0 = -1, \tag{53}$$

$$x_{n+1} = -\frac{1}{2}A_n - \frac{1}{8}B_n, \quad n \geq 0 \tag{54}$$

We approximate x by $\Omega_k = \sum_{n=0}^{k-1} x_n$ gives

$$\Omega_1 = x_0 = -1, \tag{55}$$

$$\Omega_2 = x_0 + x_1 = -1.375, \tag{56}$$

$$\Omega_3 = x_0 + x_1 + x_2 = -1.609375, \tag{57}$$

Applying the Shanks transform (13) to the sequence $\{\Omega_k\}$ for $k = 1, 2, 3$, we obtain the data given in Table 5.

n	Ω_n	A_n
1	-1.000000	
2	-1.375000	-2
3	-1.609375	

Table 5: *The results of applying Shanks transform on* $\Omega_n, n = 1, 2, 3$

As can be seen in the last column of Table 5, the obtained solution equals to -2.

5 Conclusion

In this work, we successfully applied the Shanks transformation on the ADM to improve the accuracy of the approximate solutions of ADM. Finally, we have some examples to illustrate these considerations. Numerical results of these methods are given briefly in the following table.

equation	ADM	Shanks transform	exact solution
$x^2 - (1-x)^5 = 0$	0.3454320000000	0.3456910000000	0.3459550000000
$x^3 - 7x^2 + 14x - 6 = 0$	0.5857540000000	0.5857860000000	0.5857860000000
$e^{-x} - x + 2 = 0$	2.1200282389945	2.1200282389886	2.1200282389876
$x^2 - e^x - 3x + 2 = 0$	0.2566160000000	0.2572210000000	0.2575300000000
$x^3 + 4x^2 + 8x + 8 = 0$	-1.609375000000	-2.000000000000	-2.000000000000

Table 6: *The results of the ADM, Shank transform on the ADM and exact solution*

Obviously, the Shanks transform is an efficient relation that can accelerate the convergence rate of the series.

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