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A Laplace Type Problem for Regular Lattices with Convex-Concave Cell and Obstacles Rhombus

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Abstract

In this paper we consider two regular lattices with the cell represented in the figure 1, and we compute the probability that a segment of random position and of costant lenght intersects a side of lattice. In particular we obtain the probability determinated in the previous work, then the Laplace probability.

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Cell with obstacles rhombus.

Let $\Re_1(a, b, m; \alpha)$ be the regular lattice with the fundamental cell $C_0^{(1)}$ is represented in the figure

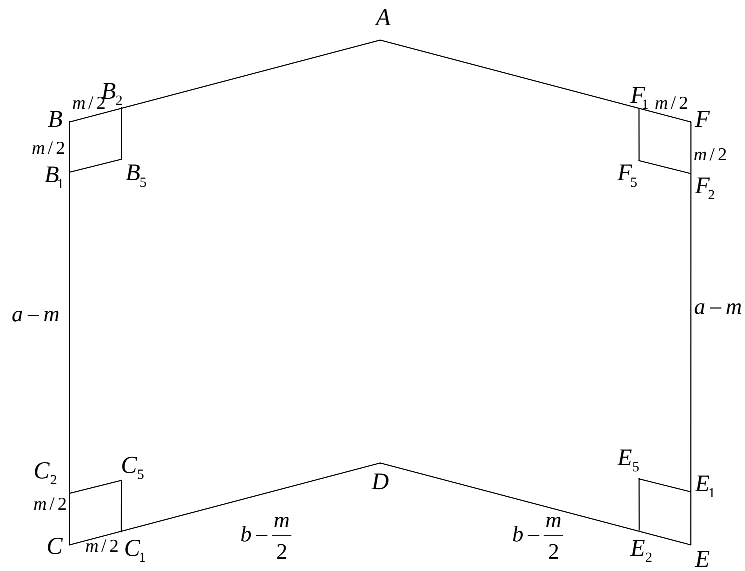

fig.1

where $m < (a, b)$ and $\alpha \leq \dfrac{\pi}{2}$ an angle. The obstacles are rhombus of two different types.

We have:

$$area\ C_0^{(2)} = 2ab\sin\alpha - m^2\sin\alpha. \tag{1}$$

Considering a segment s of random position and of constant leght l with $l < \min\left(a - m, b - \dfrac{m}{2}\right)$ and we compute the probabiltiy $P_{int}^{(1)}$ that this segment intersects a side of lattice, then the probability that the segment s intersects the side of the fundamental cell $C_0^{(1)}$.

The position of the segment s is determinated by his middle point O and by the angle φ that the segment forms with the side CD of the fundamental cell $C_0^{(1)}$.

To compute the probability $P_{int}^{(1)}$ we consider the limit positions of the segment s for a determinated value of φ and let $\widehat{C}_0^{(1)}(\varphi)$ be the determinated figure from these positions (fig. 2):

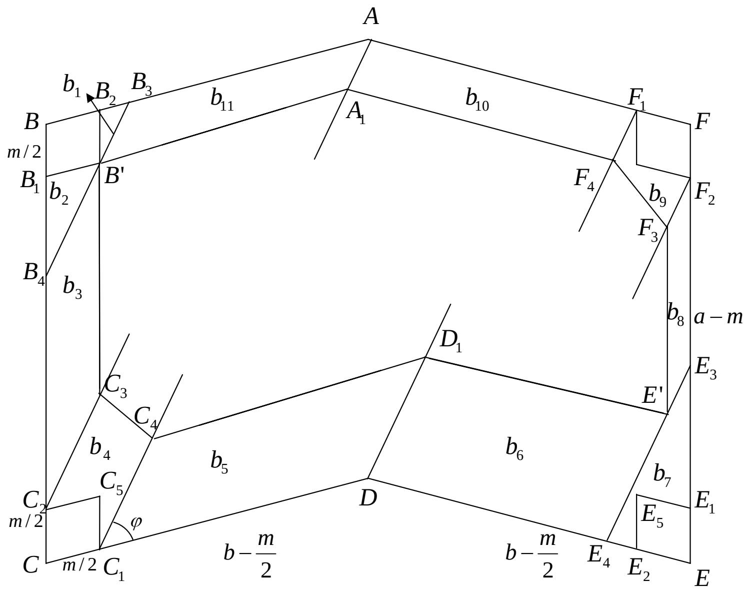

fig.2

From this figure we can write:

$$area\widehat{C}_0^{(1)}(\varphi) = areaC_0^{(1)} +$$

$$-\left[areab_1(\varphi) + areab_2(\varphi) + ... + areab_{11}(\varphi)\right]. \qquad (2)$$

We have:

$$|C_1C_2| = |E_1E_2| = m\sin\frac{\alpha}{2}, \qquad |B_1B_2| = |F_1F_2| = m\cos\frac{\alpha}{2}. \qquad (3)$$

From the figure

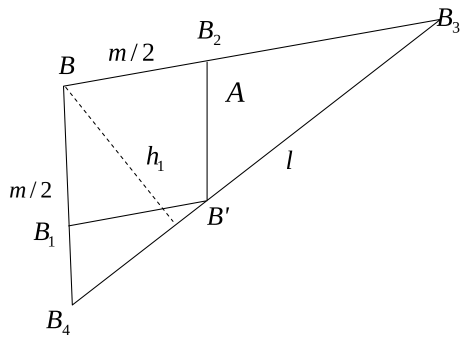

fig.3

follow that:

$$\widehat{B_1BB_2} = \pi - \alpha, \qquad \widehat{BB_3B_4} = \varphi, \qquad \widehat{BB_4B_3} = \alpha - \varphi.$$

From the triangle BB_3B_4 we have

$$\frac{l}{\sin\alpha} = \frac{|BB_4|}{\sin\varphi} = \frac{|BB_3|}{\sin(\alpha - \varphi)},$$

hence

$$|BB_3| = \frac{l\sin(\alpha - \varphi)}{\sin\alpha}, \qquad |BB_4| = \frac{l\sin\varphi}{\sin\alpha}$$

and, as $h_1 = |BB_3|\sin\varphi$, we have

$$areaBB_3B_4 = \frac{lh_1}{2} = \frac{l^2\sin\varphi\sin(\alpha - \varphi)}{2\sin\alpha}. \tag{4}$$

Then, as

$$areaBB_1B'B_2 = \frac{m^2}{4}\sin\alpha,$$

we have

$$areab_1(\varphi) + areab_2(\varphi) = \frac{l^2\sin\varphi\sin(\alpha - \varphi)}{2\sin\alpha} - \frac{m^2\sin\alpha}{4}. \tag{5}$$

The figure:

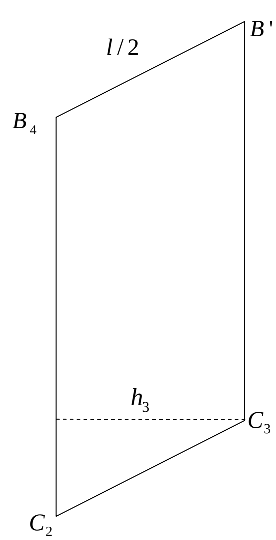

fig.4

$$\widehat{B_4C_2C_3} = \widehat{BB_4B_3} = \alpha - \varphi,$$

then

$$h_3 = \frac{l}{2} \sin \ (\alpha - \varphi) .$$

Moreover we have

$$|B_4C_2| = a - \frac{m}{2} - |BB_4| = a - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha}$$

and then

$$areab_3 (\varphi) = \left(a - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha} \right) \frac{l}{2} \sin (\alpha - \varphi) . \tag{6}$$

To compute $areab_4 (\varphi)$, we consider the figure

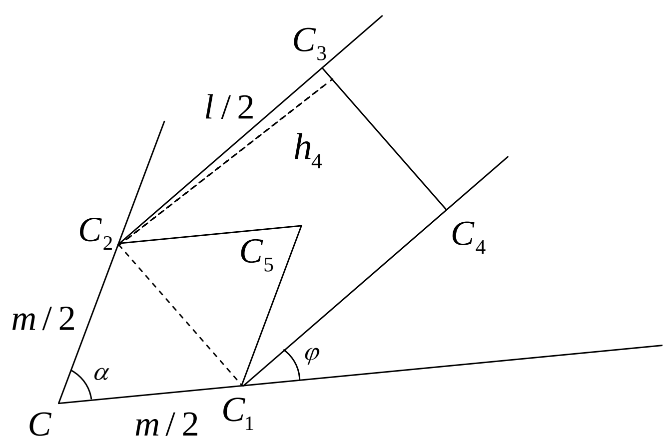

fig.5

We have:

$$\widehat{C_2C_1C_4} = \pi - \left(\frac{\pi}{2} - \frac{\alpha}{2} + \varphi \right) = \frac{\pi}{2} + \frac{\alpha}{2} - \varphi,$$

hence

$$h_4 = \frac{l}{2} \sin \ \widehat{C_2C_1C_4} = \frac{l}{2} \cos \left(\frac{\alpha}{2} - \varphi \right)$$

Then, considering the relation (3), follow that

$$areaC_1C_2C_3C_4 = |C_1C_2| \cdot h_4 = \frac{ml}{2} \sin\frac{\alpha}{2} \cos\left(\frac{\alpha}{2} - \varphi\right). \tag{7}$$

Moreover we have

$$areaC_1C_2C_5 = \frac{m^2}{8} \sin\alpha \tag{8}$$

and then

$$areab_4(\varphi) = \frac{ml}{2} \sin\frac{\alpha}{2} \cos\left(\frac{\alpha}{2} - \varphi\right) - \frac{m^2}{8} \sin\alpha. \tag{9}$$

Replacing α with $\pi - \alpha$, the figure $b_4(\varphi)$ diventa $b_9(\varphi)$, hence

$$areab_9 = \frac{ml}{2} \cos\frac{\alpha}{2} \sin\left(\frac{\alpha}{2} + \varphi\right) - \frac{m^2}{8} \sin\alpha. \tag{10}$$

Considering now the figure:

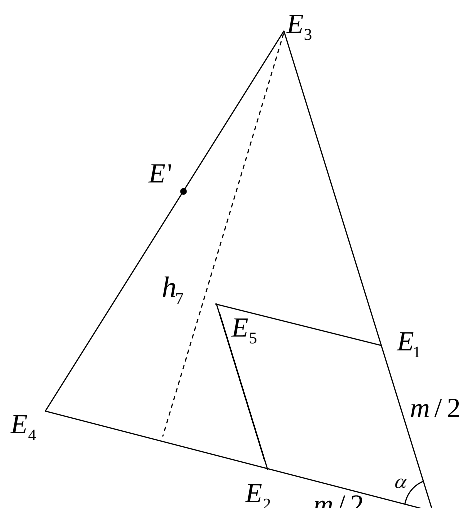

fig.6

We have $\widehat{C_4D_1E'} = 2\alpha$ and $\widehat{C_4D_1D} = \varphi$, hence

$$\widehat{DD_1E'} = 2\alpha - \varphi, \qquad \widehat{D_1DE_4} = \pi - 2\alpha + \varphi \tag{11}$$

and, then

$$\widehat{E_3E_4E} = \pi - 2\alpha + \varphi$$

and

$$\widehat{E_4E_3E} = \alpha - \varphi. \quad (12)$$

The triangle EE_3E_4 give us

$$\frac{l}{\sin\alpha} = \frac{|EE_4|}{\sin(\alpha-\varphi)} = \frac{|EE_3|}{\sin(2\alpha-\varphi)},$$

therefore

$$|EE_3| = \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}, \qquad |EE_4| = \frac{l\sin(\alpha-\varphi)}{\sin\alpha}. \quad (13)$$

We have

$$h_7 = l\sin\widehat{E_3E_4E} = l\sin(2\alpha-\varphi).$$

hence

$$areaEE_3E_4 = \frac{1}{2}|EE_4|\cdot h_7 = \frac{l^2\sin(\alpha-\varphi)\sin(2\alpha-\varphi)}{2\sin\alpha}. \quad (14)$$

From here and from (8) follow that

$$areab_7(\varphi) = \frac{l^2\sin(\alpha-\varphi)\sin(2\alpha-\varphi)}{2\sin\alpha} - \frac{m^2}{8}\sin\alpha. \quad (15)$$

The figure

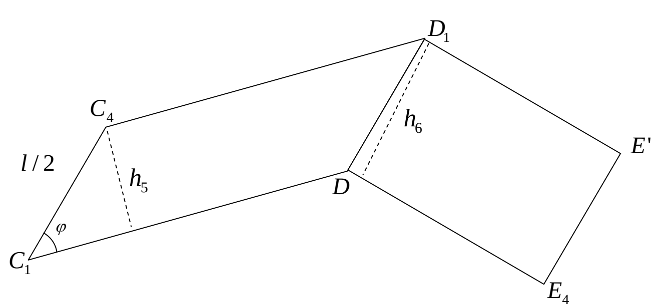

fig.7

give us $h_5 = \frac{l}{2}\sin\varphi$ and, then $|C_1D| = b - \frac{m}{2}$, we have

$$areab_5(\varphi) = \left(b - \frac{m}{2}\right)\frac{l}{2}\sin\varphi. \tag{16}$$

From the same figure 7 follow that

$$areab_6(\varphi) = |DE_4| \cdot h_6$$

Considering the relation (11) we can write:

$$h_6 = \frac{l}{2}\sin\widehat{D_1DE_4} = \frac{l}{2}\sin(2\alpha - \varphi).$$

Then, with the (13),

$$|DE_4| = b - |EE_4| = b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}$$

and therefore

$$areab_6(\varphi) = \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(2\alpha-\varphi). \tag{17}$$

The figure

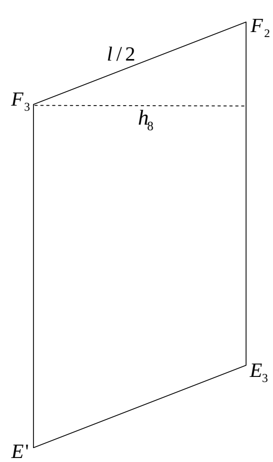

fig.8

give us the possibility to compute $areab_8(\varphi)$.

Considering the relation (12), we have

$$\widehat{E_3F_2F_3} = \widehat{E_4E_3E} = \alpha - \varphi,$$

then

$$h_8 = \frac{l}{2}\sin\widehat{E_3F_2F_3} = \frac{l}{2}\sin(\alpha-\varphi).$$

From the relation (13) follow that

$$|E_3F_2| = a - \frac{m}{2} - |EE_3| = a - \frac{m}{2} - \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}.$$

Hence

$$areab_8(\varphi) = \left[a - \frac{m}{2} - \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(\alpha-\varphi). \qquad (18)$$

We cosidered the figure

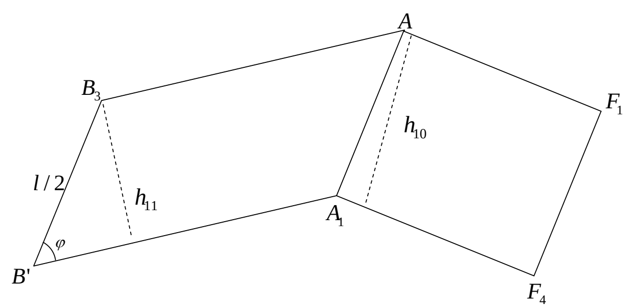

fig.9

and, considering the (11), we have

$$\widehat{AA_1F_4} = \widehat{D_1DE_4} = \pi - 2\alpha + \varphi.$$

Then, considering the figure 3, follow that:

$$|AB_3| = b - |BB_3| = b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}.$$

Moreover,we have:

$$|AF_1| = b - \frac{m}{2}, \quad h_{10} = \frac{l}{2}\sin(2\alpha - \varphi), \quad h_{11} = \frac{l}{2}\sin\varphi.$$

Therefore

$$areab_{10} = \left(b - \frac{m}{2}\right) \cdot \frac{l}{2}\sin(2\alpha - \varphi) \tag{19}$$

and

$$areab_{11}(\varphi) = \left[b - \frac{l\sin(\alpha - \varphi)}{\sin\alpha}\right] \cdot \frac{l}{2}\sin\varphi. \tag{20}$$

Replacing in the (2) the expression (5), (6), (9), (10), (11), (13), (14), (15), and (20), we obtain

$$area\widehat{C}_0^{(1)}(\varphi) = areaC_0^{(1)} -$$

$$\left\{\frac{l}{2}\left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha\right]\cos\varphi + \right.$$

$$+\frac{l}{2}\left[2b + \frac{m}{2} - (2a - m)\cos\alpha - \left(2b - \frac{m}{2}\right)\cos 2\alpha\right] \cdot$$

$$\left. \sin\varphi - \frac{l^2}{2}\sin(2\alpha - \varphi) - \frac{5m^2\sin\alpha}{8}\right\}. \tag{21}$$

Denoting with M_1 the set of segments s that have the middle point in the fundamental cell $C_0^{(1)}$and with N_1 the set of segments s completely contained in $C_0^{(1)}$, we have that :

$$P_{int}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)}. \tag{22}$$

As in the previous paragraph we have $\varphi \in [0, \alpha]$.

Therefore

$$\mu\left(M_1\right)=\int_0^{\alpha} d\varphi \iint\limits_{\left\{(x,y)\in C_0^{(1)}\right\}} dxdy=\int_0^{\alpha}\left[areaC_0^{(1)}\right]d\varphi=\alpha areaC_0^{(1)} \tag{23}$$

and, considering the (21),

$$\mu\left(N_1\right)=\int_0^{\alpha} d\varphi \iint\limits_{\left\{(x,y)\in \widehat{C}_0^{(1)}(\varphi)\right\}} dxdy=\int_0^{\alpha}\left[area\ \widehat{C}_0^{(1)}\left(\varphi\right)\right]d\varphi=$$

$$\alpha areaC_0^{(1)}-\left\{\frac{l}{2}\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\sin\varphi-\right.$$

$$\frac{l}{2}\left[2b+\frac{m}{2}-(2a-m)\cos\alpha-\left(2b-\frac{m}{2}\right)\cos 2\alpha\right]\cdot$$

$$\left.\cos\varphi-\frac{l^2}{4}\cos 2\left(\alpha-\varphi\right)-\frac{5m^2\sin\alpha}{8}\varphi\right\}\Bigg|_0^{\alpha}=$$

$$\alpha areaC_0^{(1)}-\left(\left\{\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\sin\alpha+\right.\right.$$

$$+\left[2b+\frac{m}{2}-(2a-m)\cos\alpha-\left(2b-\frac{m}{2}\cos 2\alpha\right)\right]\cdot$$

$$\left.\left.(1-\cos\alpha)\right\}\frac{l}{2}-\frac{1-\cos 2\alpha}{4}l^2-\frac{5m^2\alpha\sin\alpha}{8}\right). \tag{24}$$

The relation (1), (22), (23) and (24) give us:

$$P_{int}^{(1)}=\frac{1}{\alpha\left(2ab\sin\alpha-m^2\sin\alpha\right)}\left(\left\{\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\sin\alpha+\right.\right. \tag{25}$$

$$\left[2b+\frac{m}{2}-(2a-m)\cos\alpha-\left(2b-\frac{m}{2}\cos 2\alpha\right)\right](1-\cos\alpha)\right\}\frac{l}{2}$$

$$-\frac{1-\cos 2\alpha}{4}l^2-\frac{5m^2\alpha\sin\alpha}{8}\Bigg).$$

For $\alpha=\dfrac{\pi}{2}$, the fundamental cell began a rectangle with sides a and $2b$ and with four square obstacles with side $\dfrac{m}{2}$ and the probability (25) began:

$$P_1=\frac{2\left(a+2b\right)l-l^2-\dfrac{5\pi m^2}{8}}{\pi\left(2ab-m^2\right)}$$

Evidentement for $m\to 0$ we have the Laplace probability:

$$P=\frac{2\left(a+2b\right)l-l^2}{2\pi ab}.$$

2 Cell with obstacles rhombus and circular sections.

Let $\Re_2\left(a,b,m;\alpha\right)$ be the regular lattice with the fundamental cell $C_0^{(2)}$ is represented in the figure

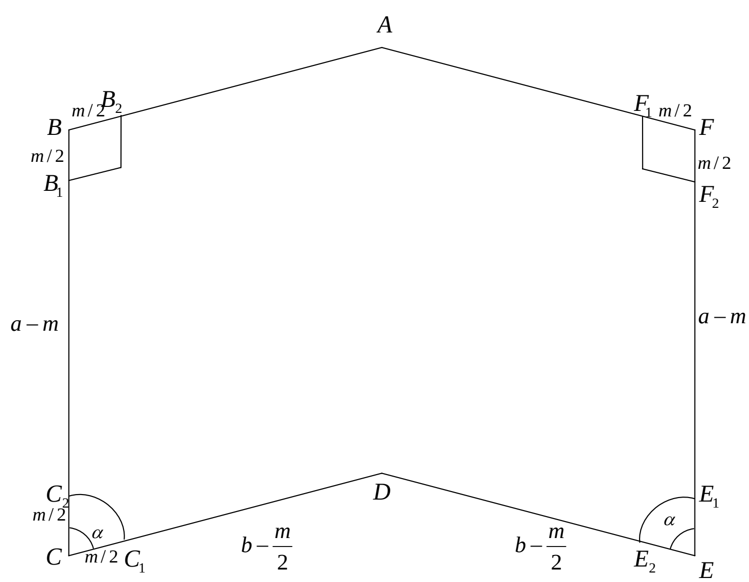

fig.10

where $m < \min(a, b)$ and $\alpha \leq \dfrac{\pi}{2}$ an angle. The four obstacles are two rhombus and two circular sections.

We have:

$$area\ C_0^{(4)} = 2ab\sin\alpha - \frac{m^2\sin\alpha}{2} - \frac{\pi m^2}{8}. \tag{26}$$

Considering a segment s of random position and of constant leght l with $l < \min\left(a - m,\ b - \dfrac{m}{2}\right)$and we compute the probabiltiy $P_{int}^{(2)}$ that this segment intersects a side of lattice, then the probability that the segment s intersects the side of the fundamental cell $C_0^{(2)}$.

The position of the segment s is determinated by his middle point O and by the angle φ that the segment forms with the side CD of the fundamental cell $C_0^{(2)}$.

To compute the probability $P_{int}^{(2)}$ we consider the limit positions of the segment s for a determinated value of φ let $\widehat{C}_0^{(2)}(\varphi)$ be the determinated figure from these positions (fig. 11):

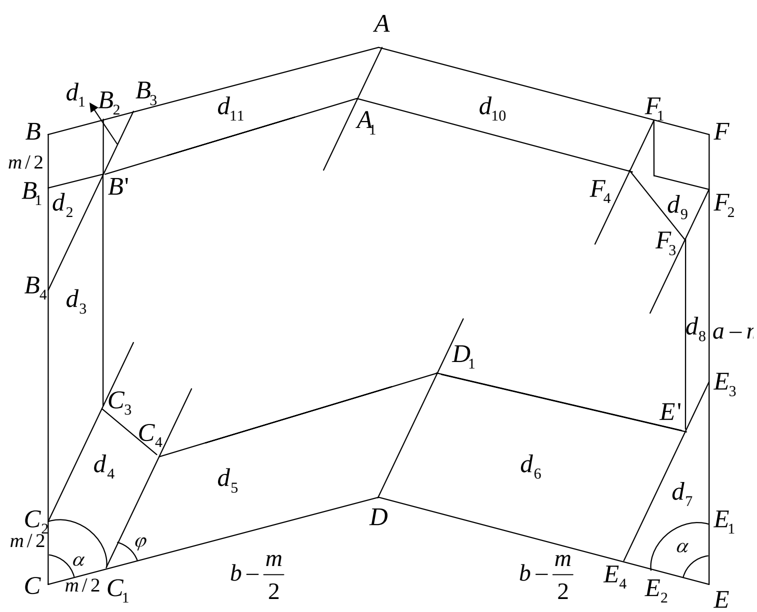

fig.11

From here we can write:

$$area\widehat{C}_0^{(2)}(\varphi) = areaC_0^{(2)} -$$

$$[aread_1(\varphi) + aread_2(\varphi) + ... + aread_{11}(\varphi)]\,. \tag{27}$$

We have:

$$aread_1(\varphi) + aread_2(\varphi) = areab_1(\varphi) + areab_2(\varphi) = \frac{l^2 \sin\varphi \sin(\alpha-\varphi)}{2\sin\alpha} - \frac{m^2 \sin\alpha}{4},$$

$$aread_3(\varphi) = areab_3(\varphi) = \left(a - \frac{m}{2} - \frac{l\sin\varphi}{\sin\alpha}\right)\frac{l}{2}\sin(\alpha-\varphi),$$

$$aread_5(\varphi) = areab_5(\varphi) = \left(b - \frac{m}{2}\right)\frac{l}{2}\sin\varphi,$$

$$aread_6(\varphi) = areab_6(\varphi) = \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\frac{l}{2}\sin(2\alpha-\varphi),$$

$$aread_9(\varphi) = areab_9(\varphi) = \frac{ml}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}+\varphi\right) - \frac{m^2}{8}\sin\alpha,$$

$$aread_{10}(\varphi) = areab_{10}(\varphi) = \left(b - \frac{m}{2}\right)\frac{l}{2}\sin(2\alpha-\varphi),$$

$$aread_{11}(\varphi) = areab_{11}(\varphi) = \left[b - \frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\frac{l}{2}\sin\varphi,$$

$$aread_4(\varphi) = areac_4(\varphi) = \frac{ml}{2}\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\varphi\right) - \frac{m^2(\alpha-\sin\alpha)}{8},$$

$$aread_7(\varphi) = areac_7(\varphi) = \frac{l^2\sin(2\alpha-\varphi)\sin(\alpha-\varphi)}{2\sin\alpha} - \frac{\alpha m^2}{8},$$

$$aread_8(\varphi) = areab_8(\varphi) = areac_8(\varphi) =$$

$$\left[a - \frac{m}{2} - \frac{l\sin(2\alpha-\varphi)}{\sin\alpha}\right]\cdot\frac{l}{2}\sin(\alpha-\varphi). \tag{28}$$

Replacing the (27) expression (28) , we obtain

$$area\widehat{C}_0^{(2)}(\varphi)=areaC_0^{(2)}-\left\{\frac{l^2\sin\varphi\sin(\alpha-\varphi)}{2\sin\alpha}-\frac{m^2\sin\alpha}{4}+\right.$$

$$+\left(a-\frac{m}{2}-\frac{l\sin\varphi}{\sin\alpha}\right)\ \frac{l}{2}\sin(\alpha-\varphi)+\left(b-\frac{m}{2}\right)\frac{l}{2}\sin\varphi+$$

$$+\left[b-\frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\cdot\ \frac{l}{2}\sin(2\alpha-\varphi)+\frac{ml}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\alpha}{2}+\varphi\right)+$$

$$-\frac{m^2\sin\alpha}{8}+\left(b-\frac{m}{2}\right)\frac{l}{2}\sin(2\alpha-\varphi)+\left[b-\frac{l\sin(\alpha-\varphi)}{\sin\alpha}\right]\frac{l}{2}\sin\varphi+$$

$$\frac{ml}{2}\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\varphi\right)-\frac{m^2(\alpha-\sin\alpha)}{8}+$$

$$+\frac{l^2\sin(2\alpha-\varphi)\sin(\alpha-\varphi)}{2\sin\alpha}-\frac{\alpha m^2}{8}+$$

$$\left.+\left[a-\frac{m}{2}-\frac{l\sin(2\alpha-\varphi)}{\sin\alpha}\right]\cdot\ \frac{l}{2}\sin(\alpha-\varphi)\right\}=$$

$$areaC_0^{(2)}-\left\{\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\frac{l}{2}\cos\varphi+\right.$$

$$+\left[2b+\frac{m}{2}-(2a-m)\cos\alpha-\left(2b-\frac{m}{2}\right)\cos 2\alpha\right]\frac{l}{2}\sin\varphi+$$

$$\left.-l^2\cdot\frac{\sin\varphi\sin(\alpha-\varphi)}{\sin\alpha}-\frac{m^2}{8}(\alpha+2\sin\alpha)\right\}. \tag{29}$$

Denoting with M_2 the set of segments s that have the middle point in the fundamental cell $C_0^{(2)}$and with N_2 the set of segments s completely contained in the fundamental cell $C_0^{(2)}$, we have that :

$$P_{int}^{(2)}=1-\frac{\mu(N_2)}{\mu(M_2)}, \tag{30}$$

We have

$$\mu\left(M_2\right)=\int\limits_0^{\alpha}d\varphi\iint\limits_{\left\{(x,y)\in C_0^{(2)}\right\}}dxdy=\int\limits_0^{\alpha}\left[areaC_0^{(2)}\right]d\varphi=\alpha areaC_0^{(2)} \tag{31}$$

and, considering the (29),

$$\mu\left(N_2\right)=\int\limits_0^{\alpha}d\varphi\iint\limits_{\left\{(x,y)\in\widehat{C}_0^{(2)}(\varphi)\right\}}dxdy=\int\limits_0^{\alpha}\left[areaC_0^{(2)}\left(\varphi\right)\right]d\varphi=$$

$$\alpha areaC_0^{(2)}-\left\{\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\frac{l}{2}\sin\varphi-\right.$$

$$-\left[2b+\frac{m}{2}-\left(2a-m\right)\cos\alpha-\left(2b-\frac{m}{2}\right)\cos 2\alpha\right]\frac{l}{2}\sin\varphi+$$

$$\left.+\frac{l^2}{2\sin\alpha}\left[\frac{\sin\left(\alpha-2\varphi\right)}{2}+\varphi\cos\alpha\right]-\frac{m^2\varphi}{2}\left(\pi-\sin\alpha\right)\right\}\Bigg|_0^{\alpha}=$$

$$\alpha areaC_0^{(2)}-\left\{\left[2a\sin\alpha+\left(2b-\frac{m}{2}\right)\sin 2\alpha\right]\frac{l}{2}\sin\alpha+\right.$$

$$\left[2b+\frac{m}{2}-\left(2a-m\right)\cos\alpha-\left(2b-\frac{m}{2}\right)\cos 2\alpha\right]\frac{l}{2}\left(1-\cos\alpha\right)$$

$$\left.-\frac{l^2}{2}\left(1-\alpha ctg\alpha\right)-\frac{m^2\alpha}{2}\left(\pi-\sin\alpha\right)\right\}. \tag{32}$$

The relation (26), (30), (31) and (32) give us:

$$P_{int}^{(2)}=\frac{1}{\alpha\left(2ab\sin\alpha-\dfrac{m^2\sin\alpha}{2}-\dfrac{\pi m^2}{8}\right)}\cdot$$

$$\left\{ \left[2a\sin\alpha + \left(2b - \frac{m}{2}\right)\sin 2\alpha\right]\frac{l}{2}\sin\alpha + \right.$$

$$\left[2b + \frac{m}{2} - (2a-m)\cos\alpha - \left(2b - \frac{m}{2}\right)\cos 2\alpha\right]\frac{l}{2}(1-\cos\alpha)$$

$$\left. -\frac{l^2}{2}(1-\alpha ctg\alpha) - \frac{m^2\alpha}{2}(\pi - \sin\alpha)\right\}. \tag{33}$$

For $\alpha = \frac{\pi}{2}$ and $m = 0$, the fundamental cell began a rectangle with side a and $2b$ and the probability (33) began the Laplace probability:

$$P = \frac{2(a+2b)l - l^2}{2\pi ab}.$$

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